MANUAL

OF

ALGEBRA.

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PREFACE.

THE following manual was prepared for the use of the students of Columbia College, and in its original form it has been employed as a text-book, not only in that institution, but in various Colleges, Academies, High Schools, and other institutions of learning. The flattering manner in which it has been received by our most successful teachers of Mathematics, has induced the Author to publish it in its present revised form.

In preparing it anew for the press, such alterations and improvements have been made as have been suggested by the author's practical experience in its use as a college text-book. The opening chapters have been somewhat simplified, the chapter on logarithms has been extended, a section on inequalities has been added, and the whole has been carefully corrected and revised. It is hoped that these modifications will better adapt it to meet the wants of academies and high schools, without in any way impairing its value as a college book.

The original design of the work was to bring the methods of Bourdon within the reach of those who had not the time, perhaps not the inclination, to study the more extended and complete work of that eminent algebraist. In the present edition no attempt has been made to modify that plan; on the contrary it is believed that every change made has been in the direction of a closer resemblance to the great work whose spirit it would aspire to imitate.

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MANUAL OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND EXPLANATION OF SIGNS.

Definitions.

1. Quantity is anything that can be measured, as number, time, or distance.

A thing can be *measured* when it can be expressed in terms of some other thing of the same kind taken as a *unit*.

The **value** of a quantity is an expression for that quantity in terms of some assumed unit; as 7 *feet*, 3 *years*, 4 *pounds*.

2. Mathematics is the science that treats of the relations of quantities, and of the operations that may be performed on them.

3. Algebra is a branch of Mathematics in which quantities to be considered are represented by letters, and operations to be performed on them are indicated by signs.

The letters and signs are called *symbols*.

In what follows, the expressions 1°, 2°, 3°, &c., stand for *first*, *second*, *third*, &c.

Explanation of Symbols.

4. The quantities treated of in Algebra are of two kinds:

1°. *Known quantities,* those whose values are given; and,

2°. *Unknown quantities,* those whose values are required.

Known quantities are generally represented by leading letters of the alphabet; as, a, b, c, &c.

Unknown quantities are generally represented by final letters of the alphabet; as, x, y, z, w, &c.

When, in the course of an operation, an unknown quantity becomes known, it is often convenient to represent it by one of the final letters, with one or more accents, as, x', y'', z''', &c. These symbols are read, *x prime, y second, z third,* &c.

5. The signs employed in Algebra are of three kinds: signs of operation; signs of relation; and signs of abbreviation.

The signs of operation are the following:

1°. The *sign of addition,* $+$, called **plus**. When placed between two quantities, it indicates that the second is to be added to the first. Thus, the expression, $a + b$, read, *a plus b,* indicates that b is to be added to a.

2°. *Sign of subtraction,* $-$, called **minus**. When placed between two quantities, it indicates that the second is to be subtracted from the first. Thus, the expression, $c - d$, read *c minus d*, indicates that d is to be subtracted from c.

The *double sign*, ±, read *plus and minus*, is used to indicate that the quantity before which it is placed, is first to be added to, and then to be subtracted from, the preceding quantity. Thus, the expression $a \pm b$ is equivalent to the two expressions, $a + b$, and $a - b$.

If no sign is written before a quantity, the sign + is understood.

3°. The *sign of multiplication*, ×. When placed between two quantities it indicates that one of them is to be multiplied by the other. Thus, the expression $x \times y$ indicates that x is to be multiplied by y, or that y is to be multiplied by x. The quantities x and y are called **factors**, and the result of the multiplication is called a **product**. If more than two factors are multiplied together, the result is called a **continued product**.

Factors represented by letters are called *literal factors*; in this case the sign of multiplication may be replaced by a simple dot, or it may be omitted altogether. Thus, the continued product of x, y, and z, may be represented by any one of the expressions

$$x \times y \times z, \qquad x \,.\, y \,.\, z, \qquad \text{or} \qquad x\,y\,z.$$

4°. The *sign of division*, ÷. When written between two quantities, it indicates that the first is to be divided by the second. Thus, the expression, $p \div q$, indicates that p is to be divided by q. The operation may also be expressed by writing one quantity over the other, in the form of a fraction; or the sign of division may be replaced either by a straight, or by a curved line. Thus, the quotient of p by q may be represented by any one of the expressions,

$$p \div q, \quad \frac{p}{q}, \quad p\underline{|q}, \quad \text{or} \quad q)p.$$

5°. The *exponential sign.* The exponential sign is a number written on the right, and above a quantity, to show how many times that quantity is to be taken as a factor. Thus, in the expressions x^2, x^4, x^m, the numbers 2, 4, and m, are **exponents**, indicating respectively that x is to be taken 2, 4, and m times, as a factor.

The resulting products are called **powers**. Thus, x^4 is called the fourth power of x.

If no exponent is written, the exponent 1 is always understood.

6°. The *radical sign,* $\sqrt{}$. When placed over a quantity, it indicates that a root of that quantity is to be extracted. The nature of the root is indicated by a number placed over the radical sign, called an **index**. Thus, the expressions, $\sqrt{a}$, $\sqrt[3]{a}$, and $\sqrt[n]{a}$, indicate that the *square, cube,* and *nth roots* of a, are to be extracted.

If no index is written, the index 2 is always understood.

The signs of relation are the following:

1°. The *sign of equality,* $=$. When written between two quantities, it indicates that they are equal to each other. Thus, the expression, $a = nb$, indicates that a is equal to the product of n and b.

2°. The *sign of inequality,* $<$, $>$. When written between two quantities, it indicates that they are unequal, the greater one being at the opening of the sign. Thus, the expressions, $a > b$, and $c < d$, indicate that *a is greater than b*, and that *c is less than d.*

3°. The *signs of proportion*, : :: :. The single colon stands for, *is to;* the double colon for, *as.* Thus, the expression,

$$a : b :: c : d,$$

is read, *a is to b, as c is to d.*

The double colon is equivalent to the sign of equality and is often replaced by that sign. Thus, the preceding proportion may be written,

$$a : b = c : d.$$

The signs of abbreviation are the following:

1°. The sign ∴, stands for the word *hence.*

2°. The *vinculum*, ———, the *bar*, |, and the *parenthesis*, or *brackets*, ( ), [ ], $\Big\{\ \Big\}$, are used to connect several quantities, which are to be operated on as a single quantity. Thus, each of the expressions,

$$\overline{a+b} \times x, \qquad \begin{array}{r|l} a & x, \\ +b & \end{array} \quad \text{and} \ (a+b)x,$$

indicates that the *sum* of a and b is to be multiplied by x.

Other signs will be explained in their proper places.

Additional Meaning of the Signs + and −.

6. The signs + and −, besides indicating addition and subtraction, are also used to show the *sense* in which a quantity is taken:

A quantity preceded by the sign + is said to be *positive,* or *additive;* a quantity preceded by the sign − is said to be *negative,* or *subtractive.* If we agree to call a quantity *positive* when taken in any sense, we must call it *negative* when taken in an opposite sense. If we agree to call distances measured toward the right *positive,* distances measured toward the left must be *negative;* hence, the sign − written before a quantity changes the *sense* in which the quantity is to be taken, that is, *it changes a positive quantity to a negative one, and a negative quantity to a positive one.* Thus $-(+a)$ is the same as $-a$, and $-(-a)$ is the same as $+a$.

Definitions.

7. A **coefficient** is a number written before a quantity to show how many times it is to be taken *additively.* Thus, the expression, $3a^2$, is equivalent to $a^2 + a^2 + a^2$. The number 3 is the coefficient of a^2. The coefficient may be either *numerical* or *literal.* Thus, in the expressions, $3x^2$, $3ax^2$, $(a + b + c)x^2$, the quantities 3, $3a$, and $(a + b + c)$, are coefficients of x^2.

If a coefficient is spoken of, without indicating its nature, we generally mean a numerical coefficient. If no coefficient is written, the coefficient 1 is always understood.

8. An **algebraic expression** is an expression for a quantity written in algebraic language, that is, by means of algebraic symbols. Thus, $2x^3 - 3y^2$, is the algebraic expression for *twice the cube of x diminished by three times the square of y.*

The parts of an expression that are connected by the signs + and — are called **terms.** Terms preceded by the sign +, either expressed or understood, are called *positive terms;* those preceded by the sign — are called *negative terms.* Thus, in the expression $3a^2-b+4c-d^2$, the terms $3a^2$ and $4c$ are positive, and the terms $-b$ and $-d^2$ are negative.

9. A **monomial** is a single term, unconnected with any other by the signs + or —. Thus, $3a^2$, $7a^2bc$, are monomials.

A monomial consists of three parts:

1°. A *literal* part, which may be regarded as the unit;

2°. A *coefficient,* which shows how many times the unit is taken; and

3°. A *sign,* which shows the sense in which it is taken.

Thus, in the monomial $-3a^2x$, we may regard the literal part, a^2x, as the *unit;* the coefficient, 3, shows that this unit is taken 3 *times;* and the sign, —, shows that it is taken in a *negative sense.*

10. A **polynomial** is a collection of terms connected by the signs + or —. Thus, $3a^2b+c-d$, $\frac{2ab-c}{3}+d$, are polynomials.

11. A **binomial** is a polynomial of two terms; as,

$$c+d, \qquad (e+f)x, \qquad \frac{a}{b}x+y.$$

12. A **trinomial** is a polynomial of three terms; as,

$$a + b + c, \qquad x^2 - 2xy + y^2, \qquad -a + 3b^3 + x.$$

Classification of Terms.

13. Terms are of different **degrees**, according to the number of literal factors they contain: those that contain but one literal factor are of the *first degree;* those that contain two literal factors are of the *second degree;* and so on. Thus, the term $3a$ is of the first degree, because it contains but one literal factor; the term $-7a^2$ is of the second degree, because it contains two literal factors; the term $-a^2bx^3$ is of the sixth degree, because it contains six literal factors.

The degree of a term is determined by the sum of the exponents of all its letters.

Two terms are **homogeneous**, when they are of the same degree. Thus, the terms $8a^2bx$ and y^3z are homogeneous, also the terms a^3b^2c and $7x^3y^3$.

A *polynomial is homogeneous* when all its terms are homogeneous. Thus, the polynomial $a^3bc - 7ax^4 + 3b^3c^2$ is homogeneous, but the polynomial $8a^3b - 7a^2c + 8x^2$ is not homogeneous.

Two terms are **similar**, or **like**, when the combination of literal factors is the same in both. Thus, the terms $8x^2yz$ and $-7x^2yz$, are similar, as are also the terms $25a^2bcd^3$ and $2a^2bcd^3$. In order that two terms may be similar, they must contain the same letters, and each letter must have the same exponent. The terms $8a^2b$ and $-7ab^2$ contain the same letters, but are not similar.

Definitions.

14. The **reciprocal** of a quantity is 1 divided by that quantity: thus, $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{c}{d}$, x, are reciprocals of the quantities a, $a+b$, $\frac{d}{c}$ and $\frac{1}{x}$. The product of any quantity by its reciprocal, is equal to 1. Thus $ab \times \frac{1}{ab}$ is equal to 1.

15. The **numerical value** of an expression, is the result obtained by assigning a numerical value to each letter that enters it, and then performing all the indicated operations. Thus, the numerical value of the expression,

$$ab + ac + bc,$$

when $a = 2$, $b = 3$, and $c = 4$, is

$$2 \times 3 + 2 \times 4 + 3 \times 4 = 26.$$

EXAMPLES.

Find the numerical values of the following expressions, when $a = 2$, $b = 3$, $c = 4$, and $d = 5$.

1. $ab + cb$. *Ans.* 18.
2. $ad - d + b$. *Ans.* 8.
3. $bc + ab - c$. *Ans.* 14.
4. $(bc + a)\,b$. *Ans.* 42.
5. $(bd - a)(ac - d)$. *Ans.* 39.
6. $(d + c)(d - c)$. *Ans.* 9.
7. $\frac{a + 2b}{c} + d$. *Ans.* 7.

8. $(a^2 - b)(c + d)$. *Ans.* 9.

9. $abc + cd + ad$. *Ans.* 54.

10. $\dfrac{a + bc}{7} \times (c + d)$. *Ans.* 18.

Find the numerical values of the following expressions, when $a = 5$, $b = 2$, $c = 4$, and $d = 3$.

11. $\dfrac{6}{a} - \dfrac{3}{b} + \dfrac{10}{c - d} - \dfrac{14}{c + d}$. *Ans.* $7\frac{7}{10}$.

12. $\left(\dfrac{a^2b}{c} \times d\right) \div \left(\dfrac{ab^2}{c} + d\right)$. *Ans.* 4.6875.

13. $\dfrac{a^2b^2c^2d^2 + 2abcd + 1}{abcd + 1}$. *Ans.* 121.

14. $\dfrac{a^2 + b^2 - d^2}{a + b + d} + \dfrac{abcd}{2b + c} - \dfrac{4a^2 - 10bc + 2}{2c + d}$.

Ans. 15.

15. $\dfrac{12(a + b^2)}{d^3} - (c - b) + \dfrac{7}{a^2 - b^2} \times \dfrac{14}{c^2 - d^2}$.

Ans. $2\frac{2}{3}$.

16. $\left\{[\overline{a + b} \times c + d]\ b + a\right\} \times c$. *Ans.* 268.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

I. ADDITION.

Definitions.

16. **Addition** is the operation of finding the simplest expression for the aggregate of two or more quantities.

This expression is called their **sum**.

Explanation.

17. If the quantities are similar, the addition may be performed; if they are not similar, the operation can only be indicated. Thus, the sum of $7a^2b$ and $4a^2b$, is $11a^2b$, in the same way that the sum of 7 pounds and 4 pounds, is 11 pounds. The sum of $3a^2c$ and $4b^2$, cannot be expressed by a single term, any more than the sum of 3 pounds and 4 feet; the sum may, however, be indicated by writing the quantities one after another, with their proper signs; thus,

$$3ac^2 + 4b^2.$$

Addition of Similar Terms.

18. To deduce a rule for adding similar terms, let us take the following examples:

(1.)	(2.)	(3.)	(4.)
$+\ 7a^2bc$	$-\ 2a^2bc$	$+\ 4a^2bc$	$-\ 8a^2bc$
$+\ a^2bc$	$-\ 3a^2bc$	$-\ 2a^2bc$	$+\ 5a^2bc$
$+\ 3a^2bc$	$-\ a^2bc$	$+\ 7a^2bc$	$-\ 2a^2bc$
$+\ 5a^2bc$	$-\ 8a^2bc$	$-\ 5a^2bc$	$+\ 3a^2bc$
$+\ 16a^2bc$	$-\ 14a^2bc$	$+\ 4a^2bc$	$-\ 2a^2bc$

In the first example the unit a^2bc is taken *positively* 5 *times*, 3 *times*, 1 *time*, and 7 *times*, that is, it is taken positively 16 *times;* hence, in this case, the sum of the monomials is $+ 16a^2bc$. In the second example the same unit is taken *negatively* 8 *times*, 1 *time*, 3 *times*, and 2 *times*, that is, it is taken negatively 14 *times;* hence, the sum of the monomials, in this case, is $- 14a^2bc$.

In the third example the unit a^2bc is taken *positively* 11 *times*, and *negatively* 7 *times*, that is, it is taken positively 4 *times* more than it is taken negatively; hence, the sum, in this case, is $+ 4a^2bc$. In the fourth example, the unit is taken *negatively* 10 *times*, and *positively* 8 *times*, that is, it is taken negatively 2 *times* more than it is taken positively; hence, the sum, in this case, is $- 2a^2bc$.

Since we may treat all similar cases in the same manner, we have the following rule for adding similar terms:

RULE.

Add the coefficients of the positive and negative terms separately; subtract the less sum from the greater, prefixing the sign of the greater; to the result annex the common literal part.

EXAMPLES.

1. Find the sum of $3ay^3$, $-5ay^3$, $-2ay^3$, and $7ay^3$.
Ans. $3ay^3$.

2. Find the sum of $4cz^4$, $-7cz^4$, $3cz^4$, and $-14cz^4$.
Ans. $-14cz^4$.

3. Find the sum of $8bc^2$, $-4bc^2$, $-11bc^2$, and $-2bc^2$.
Ans. $-9bc^2$.

4. Find the sum of ay, $-4ay$, $6ay$, and $15ay$.
Ans. $18ay$.

Addition of Polynomials.

19. The sum of two or more polynomials may be found by first writing them one after another with their proper signs, and then reducing similar terms to their simplest form by the preceding rule. In practice it is found more convenient to write the polynomials so that each group of similar terms may be found in a single column; hence, we have the following rule for the addition of polynomials:

RULE.

I. Write the quantities to be added so that similar terms shall fall in the same column.

II. Add each column of similar terms separately, and to the result annex the dissimilar terms with their proper signs.

EXAMPLES.

(1.)

$$\begin{array}{r} 3a - 3bx \\ 9a - 5bx \\ 5a - 4bx \\ \hline 17a - 12bx \end{array}$$

(2.)

$$\begin{array}{r} c + bx^2 + d \\ 4c - 2bx^2 - 2d \\ 5c + 3bx^2 \phantom{{}- 2d} \\ \hline 10c + 2bx^2 - d \end{array}$$

(3.)

$$\begin{array}{r} 3x^2y - 3y^2x - 4y + z \\ 3x^2y + 7y^2x - 8y \phantom{{}+z} \\ 8x^2y - 5y^2x + 5y \phantom{{}+z} \\ \hline 14x^2y - y^2x - 7y + z \end{array}$$

(4.)

$$\begin{array}{l} 4a + bc + 5d \\ 2a + 2bc + 3d \\ 3a - 3bc \\ \hline 9a \qquad\quad + 8d \end{array}$$

(5.)

$$\begin{array}{r} 4cx^2 + 5dy^2 - 2z^3 + d \\ 3cx^2 - 2dy^2 - 2z^3 - d \\ -2cx^2 - dy^2 + 5z^3 \phantom{{}- d} \\ \hline 5cx^2 + 2dy^2 + z^3 \phantom{{}- d} \end{array}$$

(6.)

$$\begin{array}{r} 4ab - 4c + 2(a + b) \\ 3ab + 5c + 5(a + b) \\ ab + c + 3(a + b) \\ -2ab + 7c - 4(a + b) \\ -ab - c - 2(a + b) \\ \hline 5ab + 8c + 4(a + b) \end{array}$$

(7.)

$$\begin{array}{r} 12x^2y + 2(a + b)z^2 \\ -11x^2y - (a + b)z^2 \\ 4x^2y + 4(a + b)z^2 \\ -3x^2y + 2(a + b)z^2 \\ x^2y + (a + b)z^2 \\ \hline 3x^2y + 8(a + b)z^2 \end{array}$$

Find the sums of the following groups of polynomials:

1. $a + b + c$, $a + b - c$, $a - b + c$, and $-a + b + c$.
Ans. $2a + 2b + 2c$.

2. $2ax + 3by$, $3ax + 2by$, $7ax + by$, and $8ax + 7by$.
Ans. $20ax + 13by$.

3. $2a^2 - 17ab + 3b^2$, $5a^2 + 12ab - 5b^2$, $6ab + 12a^2 - 9b^2$, and $3b^2 + 6ab + 3a^2$. *Ans.* $22a^2 + 7ab - 8b^2$.

4. $x^3 - y^3 + 2xy^2 - 3x^2y$, $2x^3 - 3xy^2 - 5x^2y + 2y^3$, $6x^2y + 6xy^2 - x^3 - y^3$, and $5xy^2 - 2y^3 - 4x^3 + 8x^2y$.
Ans. $-2x^3 + 6x^2y + 10xy^2 - 2y^3$.

5. $2x+3y-4z-10$, $8y-4x+7z+8$, $11z+5x-10y-2$, and $16 + 10x + 12y + 14z$.
Ans. $13x+13y+28z+12$.

6. $3x^3 + 2y^3 + z^3 + 8xyz$, $y^3 + 2x^3 - 3z^3 - 4xyz$, $z^3 + 3x^3 - 2y^3 - 2xyz$, and $zyx + x^3 + y^3 + z^3$.
Ans. $9x^3 + 2y^3 + 3xyz$.

7. $x^4 + 3x^3y + x^2z - 2xv$, $30x^4 - 29x^2z + 18xv - 17x^3y$, $22x^3y - 15x^4 - 32x^2z + 16xv$, and $17x^2z - 12x^4 + 6x^3y - 11xv$.
Ans. $4x^4 + 14x^3y - 43x^2z + 21xv$.

8. $ax - by$, $x - y$, $ax - x$, and $ax + x$.
Ans. $3ax + x - by - y$.

9. $ax + 2bx + 4by - 3ay$, $2ax + bx + 2ay - by$, and $4ax + 3by$.
Ans. $7ax + 3bx + 6by - ay$.

10. $px + qy + rz - c$, $2px - 2qy + 2c$, $3qy - px + 4c$, and $7px - 8qy - rz - 3c$.
Ans. $9px - 6qy + 2c$.

11. $ax^2 + a^2x - 2ax$, $x - ax + 2x^2$, $ax^2 - 2x + x^2$, and $-2ax - 2a^2x - 2ax^2$.
Ans. $3x^2 - a^2x - 5ax - x$.

12. $a^2x - ax^2 - x^2$, $ax - x^2 - a^2$, $-2a^2 - 2a^2x - 2ax^2$, and $-3a^2x + 3a^2 + 3ax^2$.
Ans. $-4a^2x - 2x^2 + ax$.

13. $a - x + 4y - 3z + w$, $z - w - y - 3a - 2x$, and $x + y + z$.
Ans. $-2a - 2x + 4y - z$.

14. $ax^2y + bxy^2z^2 + cxz^3$, $dxy^2z^2 + cxz^3$, and $2ax^2y + 4bdx$.

Ans. $3ax^2y + bxy^2z^2 + dxy^2z^2 + 2cxz^3 + 4bdx$.

Algebraic Sum.

20. The term **sum**, in algebra has a more extensive signification than in arithmetic. In arithmetic, the sum is always greater than either of the parts; in algebra, it may be numerically less than either. Thus, the sum of $7a^2b$ and $-4a^2b$, is $3a^2b$. To distinguish between these cases, the sum of two or more algebraic quantities, is called their **algebraic sum.**

II. SUBTRACTION.

Definitions.

21. Subtraction is the operation of finding from two quantities a third, which, added to the second, will give the first.

The first quantity is called the **minuend**, the second is called the **subtrahend**, and the third is called the **difference** or **remainder.**

The remainder obtained by subtracting $3a^2b$ from $5a^2b$ is $2a^2b$, because $2a^2b$ added to $3a^2b$ gives $5a^2b$. In like manner the remainder obtained by subtracting $-3a^2b$ from $5a^2b$ is $8a^2b$, because $8a^2b$ added to $-3a^2b$ gives $5a^2b$. In each of these cases the remainder is found by changing the sign of the subtrahend and adding the result to the minuend.

Rule for Subtraction.

22. To deduce a rule for subtraction, let us take the following example:

$$\begin{array}{ll}
\textit{Minuend,} & 4a^2c - 12bx^2 \\
\textit{Subtrahend,} & \underline{2a^2c - 3bx^2 + 4cy - 5z^3} \\
\textit{Remainder,} & 2a^2c - 9bx^2 - 4cy + 5z^3
\end{array}$$

In this example the subtrahend is written under the minuend so that similar terms fall in the same column; from the definition of subtraction the remainder must be

$$2a^2c - 9bx^2 - 4cy + 5z^3,$$

because this quantity added to the subtrahend gives the minuend. But this remainder can be found by changing the signs of all the terms of the subtrahend and adding the result to the minuend.

In like manner we may treat all similar cases; hence, we have the following rule for subtraction:

RULE.

I. Write the subtrahend under the minuend so that similar terms shall fall in the same column.

II. Change the signs of all the terms of the subtrahend, or conceive them to be changed, from + to —, or from — to +, and proceed as in addition.

EXAMPLES.

(1.)	(2.)	(3.)	(4.)
$12ab$	$8a^2bc$	$13a^nb$	$5a^pb^qc$
$\underline{6ab}$	$\underline{4a^2bc}$	$\underline{9a^nb}$	$\underline{2a^pb^qc}$
$6ab$	$4a^2bc$	$4a^nb$	$3a^pb^qc$

$$\begin{array}{r}(5.)\\ 7ac\\ -\ 4ac\\ \hline 11ac\end{array}\qquad\begin{array}{r}(6.)\\ 10b^2d\\ -\ 3b^2d\\ \hline 13b^2d\end{array}\qquad\begin{array}{r}(7.)\\ -\ 8a^2bc\\ +\ 3a^2bc\\ \hline -\ 11a^2bc\end{array}\qquad\begin{array}{r}(8.)\\ -\ 3a^4b^n\\ -\ 5a^4b^n\\ \hline 2a^4b^n\end{array}$$

$$\begin{array}{r}(9.)\\ 6a^2 - 8b\\ 3a^2 - 5b\\ \hline 3a^2 - 3b\end{array}\qquad\begin{array}{r}(10.)\\ 3x^3 - 4x^2y + 8\\ 5x^3 - 6x^2y - 3\\ \hline -2x^3 + 2x^2y + 11\end{array}\qquad\begin{array}{r}(11.)\\ 4xy^2 + 4z\\ -3xy^2 + 7z - 6x^3\\ \hline 7xy^2 - 3z + 6x^3\end{array}$$

12. From $2a + b - c$, subtract $a - b$.
Ans. $a + 2b - c$.

13. From $3ac - 2b$, subtract $ac - b - d$.
Ans. $2ac - b + d$.

14. From $5ab - 6$, subtract $-2ab + 6$.
Ans. $7ab - 12$.

15. From $4y^2 - 3y + 4$, subtract $2y^2 + 2y + 4$.
Ans. $2y^2 - 5y$.

16. From $219a^3 - 117a^2b + 218ab^2 + 145b^3$, subtract $26a^3 + 4a^2b + 61ab^2 - 10b^3$.
Ans. $193a^3 + 157ab^2 - 121a^2b + 155b^3$.

17. From $a - x + 2y - 3z + w$, subtract $2x + 3a - y + z - w$. *Ans.* $-2a - 3x + 3y - 4z + 2w$.

18. From $5x^3 + x^2y - 6xy^2 + y^3$, subtract $3x^3 + 4x^2y - 7xy^2 + y^3 - xy^3$. *Ans.* $2x^3 - 3x^2y + xy^2 + xy^3$.

19. From $y^4 - 4xy^3 + 7x^2y^2 - x^3y + 3x^4$, subtract $2x^4 + 3x^3y + x^2y^2 + xy^3$.
Ans. $x^4 - 4x^3y + 6x^2y^2 - 5xy^3 + y^4$.

20. From $2px^2 + ry^2 - 3qxy$, subtract $px^2 + qxy - ry^2$.
Ans. $px^2 - 4qxy + 2ry^2$.

21. From $2x^3 - 3x^2y + 2xy^2$, subtract $x^3 - xy^2 + y^3$.
Ans. $x^3 + 3xy^2 - 3x^2y - y^3$.

22. From $7x^2 - xyz + 18z$, subtract $-3x^2 - 2xyz - p - q^3$
Ans. $10x^2 + xyz + 18z + p + q^3$.

Algebraic Difference.

23. The term *difference* has a more extended signification in algebra than it has in arithmetic. In arithmetic the difference is always less than the minuend; in algebra the difference may be greater than the minuend. Thus, the difference obtained by subtracting $-4a^2$ from $6a^2$ is $10a^2$. Hence, the difference between two algebraic quantities is called the **algebraic difference**, to distinguish it from the *arithmetical difference.*

Of the Sign before a Parenthesis.

24. The sign + before a parenthesis indicates that the signs of all the included terms are to remain unchanged. In this case the parenthesis may be dropped. Thus,

$$a^3 + (c^2 - 2y) \quad = a^3 + c^2 - 2y.$$

The sign − before a parenthesis indicates that the signs of all the included terms are to be changed. In this case the parenthesis may be dropped, provided we change the signs of all the included terms. Thus,

$$a^3 - (c^2 - 2y) \quad = a^3 - c^2 + 2y.$$

Any number of the terms of a polynomial may be enclosed in a parenthesis, preceded by the sign —, provided we change the signs of all the included terms.

The following example shows some of the ways in which a polynomial may be transformed in accordance with the preceding principles·

$$\begin{aligned}
&4x^4 - 14x^3y + 4x^2z - 2w + 10, \\
&4x^4 - (14x^3y - 4x^2z + 2w - 10), \\
&4x^4 - 14x^3y - (-4x^2z + 2w - 10), \\
&4x^4 - 14x^3y + 4x^2z - (2w - 10).
\end{aligned}$$

These expressions are all equivalent, the first being the simplest.

EXAMPLES.

Reduce each of the following expressions to its simplest form:

1. $2x^3 - 3x^2y + 2xy^2 - (x^3 + y^3 - xy^2)$.
Ans. $x^3 - 3x^2y + 3xy^2 - y^3$.

2. $3x - \{x - 3a - (2y - a)\}$.
Ans. $2x + 2y + 2a$.

3. $a^2 - (b^2 - c^2) - \{b^2 - (c^2 - a^2)\} + c^2 - (b^2 - a^2)$.
Ans. $a^2 - 3b^2 + 3c^2$.

4. $x + y + z - (x - y) - (y - z) - (-y)$.
Ans. $2y + 2z$.

Essential Sign.

25. The sign that precedes the parenthesis is called the *sign of operation;* the sign that immediately precedes the quantity is called the ***sign of the quantity;*** and the sign that results from performing the indicated operation is called the **essential sign.** Thus, in the expression $a - (-b)$, which is equivalent to the expression $a + b$, the sign of the quantity b is $-$, the sign of operation is $-$, and the essential sign is $+$.

From the nature of the signs $+$ and $-$, (Art. 6), it follows that if a *positive* quantity is taken *positively*, or a *negative* quantity *negatively*, the essential sign of the result is $+$; also, if a *positive* quantity is taken *negatively*, or a *negative* quantity *positively*, the essential sign of the result is $-$.

III. MULTIPLICATION.

Definitions.

26. Multiplication is the operation of finding the product of two quantities.

The quantity to be multiplied is called the **multiplicand**, the quantity by which it is to be multiplied is called the **multiplier**, and both multiplicand and multiplier are called **factors** of the product.

The product of three or more factors is called a **continued product.**

Rule for Signs.

27. From Article 25, we deduce the following principles:

1°. If a positive quantity is taken any number of times positively, or if a negative quantity is taken any number of times negatively, the product will be positive;

2°. If a positive quantity is taken any number of times negatively, or if a negative quantity is taken any number of times positively, the product will be negative.

Hence, we have the following rule for signs:

RULE.

If two factors have like signs, their product is +, *if they have unlike signs, their product is* —.

Operation of Multiplication.

28. In algebraic multiplication, there may be three cases: 1°. both factors may be monomials; 2°. one factor may be a polynomial and the other a monomial; and 3°. both factors may be polynomials.

1°. *When both factors are monomials:*

Let it be required to find the product of $3a^2bc$ and $4a^3b^2c^4$. The product is indicated thus,

$$3a^2bc \times 4a^3b^2c^4.$$

It is shown in arithmetic that the order of the factors may be changed without affecting the value of the product. We may therefore combine the similar factors of the indicated product. The product of the factor 3 of the first monomial and the factor 4 of the second monomial is 12; the factor a is taken twice in the first monomial and three times in the second, hence it is taken 5 times in the required product; the factor b is taken once in the first monomial and twice in the second, hence it is taken 3 times in the required product; in like manner the factor c is taken 5 times in the required product; hence,

$$3a^2bc \times 4a^3b^2c^4 = 12a^5b^3c^5.$$

In like manner we may find the product of any two monomials; hence, the following rule for the multiplication of monomials:

RULE.

Multiply the coefficients together for a new coefficient: after this write all the letters in the two monomials, giving to each an exponent equal to the sum of its exponents in the two factors.

The sign of the product is determined by the rule that like signs give + and unlike signs −.

EXAMPLES.

$$\begin{array}{cccc}
(1.) & (2.) & (3.) & (4.) \\
3abc & 7a^3b^2c & 3wx^4y^3z & 8ax^4y^3z \\
4xy & 3a^2bc & 2xyz^6 & 3a^3xyz^3 \\
\hline
12abcxy & 21a^5b^3c^2 & 6wx^5y^4z^7 & 24a^4x^5y^4z^4
\end{array}$$

5. Multiply $7abc$ by $5ac$. *Ans.* $35a^2bc^2$.
6. Multiply $3ax$ by $7ac$. *Ans.* $21a^2cx$.
7. Multiply $2ay$ by $3a^2xy$. *Ans.* $6a^3xy^2$.
8. Multiply $2a^3y^2x$ by $3ax^3$. *Ans.* $6a^4y^2x^4$.

9. Multiply $12a^3x$ by $4a^3y$. *Ans.* $48a^6xy$.

10. Multiply $12a^n$ by $3a^m$. *Ans.* $36a^{m+n}$.

11. Multiply $7a^nx^m$ by $2a^nx$. *Ans.* $14a^{2n}x^{m+1}$.

The rule just given may be extended to find the continued product of any number of monomials. In this case, the sign of the product will be + when the number of negative factors is even, and − when the number of negative factors is odd.

12. Find the continued product of $8abc$, $7a^3bx$, and $3abx$.

Multiplying the coefficients 8, 7, and 3 together, we have 168, which is the coefficient of the product: adding the exponents of a, 1, 3, and 1, we have 5 for the exponent of a in the product; in like manner we find 3 for the exponent of b in the product, 1 for the exponent of c, and 2 for the exponent of x; hence, the required product is $168a^5b^3cx^2$, *Ans.*

13. Find the continued product of $-3pqr$, $-2p^2qr^3$, and $4pr^4x$. *Ans.* $24p^4q^2r^8x$.

14. Find the continued product of $-3m^pd$, $4md^2$, and $5mnd$. *Ans.* $-60m^{p+2}nd^4$.

15. Find the continued product of $-\frac{3}{4}ax^2$, $-\frac{6}{7}a^2x$, and $-\frac{1}{5}a^3x^4$. *Ans.* $-\frac{9}{70}a^6x^7$.

2°. *When one factor is a polynomial and the other a monomial:*

If each term of a polynomial is multiplied by the same quantity, the aggregate of the results will be the product of the polynomial by that quantity; hence, we have the following rule for multiplying a polynomial by a monomial:

RULE.

Multiply each term of the polynomial by the monomial, and connect the results by their proper signs.

EXAMPLES.

1. Multiply $3a^2b - 2xy + z$ by $2ax$.
Ans. $6a^3bx - 4ax^2y + 2axz$.

2. Multiply $5x^2 - 3xy + y^2$ by $-4xy$.
Ans. $-20x^3y + 12x^2y^2 - 4xy^3$.

3. Multiply $-7y^2 + 3x^3 - 2y$ by $-4x^2y$.
Ans. $28x^2y^3 - 12x^5y + 8x^2y^2$.

4. Multiply $x^n - 2ax^{n-1}y + y^q$ by $3xy^n$.
Ans. $3x^{n+1}y^n - 6ax^ny^{n+1} + 3xy^{n+q}$.

3°. *When both factors are polynomials:*

From what precedes, we have the following rule for multiplying one polynomial by another:

RULE.

Multiply every term of the multiplicand by each term of the multiplier and reduce the polynomial result to its simplest form.

EXAMPLES.

(1.)

$$\begin{array}{llll} x^2 & + 2ax & + a^2 & \\ x & + a & & \\ \hline x^3 & + 2ax^2 & + a^2x & \\ & ax^2 & + 2a^2x & + a^3 \\ \hline x^3 & + 3ax^2 & + 3a^2x & + a^3. \end{array}$$

(2.)

$$\begin{array}{llll} 2a^4b & + a^3b^2 & & \\ 2a^2b & - 2ab^2 & & \\ \hline 4a^6b^2 & + 2a^5b^3 & & \\ & - 4a^5b^3 & - 2a^4b^4 & \\ \hline 4a^6b^2 & - 2a^5b^3 & - 2a^4b^4. & \end{array}$$

It will be found convenient to *arrange* the terms of the polynomials with reference to some letter; that is, to write them down, so that the first term shall contain the highest power of that letter; the second term, the next lower power, and so on to the last term. The letter with reference to which the arrangement is made, is called the *leading letter*. In the first of the above examples, the leading letter is x; in the second, it is a. The leading letter of the product is the same as that of the factors.

3. Multiply $x^2 - xy + y^2$ by $x + y$.
Ans. $x^3 + y^3$.

4. Multiply $x^2 - xy + y^2$ by $x^2 + xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

5. Multiply $3x^2 - 2xy + 5$ by $x^2 + 2xy - 6$.
Ans. $3x^4 + 4x^3y - 4x^2y^2 - 13x^2 + 22xy - 30$.

6. Multiply $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$ by $x + y$.
Ans. $x^7 + y^7$.

7. Multiply $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$ by $x + 2y$.
Ans. $x^5 + 32y^5$.

8. Multiply $27a^2 - 13ab + 5b^2$ by $7a^2 + b^2$.
Ans. $189a^4 - 91a^3b + 62a^2b^2 - 13ab^3 + 5b^4$.

9. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
Ans. $a^3 + b^3 + c^3 - 3abc$.

10. Multiply $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ by $a - b$.
Ans. $a^5 - b^5$.

11. Multiply $2a + bc - 2b^2$ by $2a - bc + 2b^2$.
Ans. $4a^2 - b^2c^2 + 4b^3c - 4b^4$.

12. Multiply $4ab - 2ac$ by $6ab + 3ac$.
Ans. $24a^2b^2 - 6a^2c^2$.

13. Multiply $a + bx$ by $a + cx$.
Ans. $a^2 + abx + acx + bcx^2$.

14. Multiply $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^3 - 3a^2b + 3ab^2 - b^3$.
Ans. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$.

15. Multiply $a^n + 2a^mb^n + ab^p$ by $a^m - b^n$.
Ans. $a^{m+n} + 2a^{2m}b^n + a^{m+1}b^p - a^nb^n - 2a^mb^{2n} - ab^{p+n}$.

16. Multiply $x^n + y^n$ by $x^n + y^n$.
Ans. $x^{2n} + 2x^ny^n + y^{2n}$.

Find the continued product of the following groups of polynomials:

17. $x - 10$, $x + 1$, and $x + 4$.
Ans. $x^3 - 5x^2 - 46x - 40$.

18. $x - 5$, $x - 6$, $x - 7$, and $x + 8$.
Ans. $x^4 - 10x^3 - 37x^2 + 646x - 1680$.

19. $a + x$, $b + x$, and $c + x$.
Ans. $x^3 + ax^2 + bx^2 + cx^2 + abx + acx + bcx + abc$.

20. $x^2 - a^2$, $x^2 - ax + a^2$, and $x^2 + ax + a^2$.
Ans. $x^6 - a^6$.

21. $x^m - y^m$, $x^m + y^m$, and $x^n - y^n$.
Ans. $x^{2m+n} - x^ny^{2m} - x^{2m}y^n + y^{2m+n}$

IV. DIVISION.

Definitions.

29. Division is the operation of finding from two quantities a third, which multiplied by the second, will produce the first.

The first is called the **dividend**, the second the **divisor**, and the third the **quotient**.

Division is the reverse of multiplication. In multiplication we have two factors given to find the product; in division we have the product and one factor given, to find the other factor; hence, the rules for division must be the reverse of those for multiplication.

Division of Monomials.

30. Let it be required to divide $12a^5b^3c^4$ by $3a^3b^2c$. The operation can be indicated as follows:

$$\begin{array}{r} \textit{Dividend,} \\ \textit{Divisor,} \end{array} \quad \frac{12a^5b^3c^4}{3a^3b^2c} = 4a^2bc^3, \quad \textit{Quotient.}$$

The quotient must be such a monomial as multiplied by the divisor will produce the dividend; hence, its coefficient must be a number which multiplied by 3 will produce 12, and the exponent of each letter must be a number, which added to the exponent of the same letter in the divisor will give the exponent of that letter in the dividend (Art. 27). The quotient is therefore $4a^2bc^3$. Since we may treat all similar cases in the same manner, we have the following rule for dividing one monomial by another:

RULE.

Divide the coefficient of the dividend by that of the divisor for a new coefficient; after this write all the letters in the two monomials, giving to each an exponent equal to the excess of its exponent in the dividend over that in the divisor.

From the rule for signs in multiplication (Art. 27), it follows that the quotient of terms having like signs must be +, and that of terms having unlike signs must be —.

EXAMPLES.

(1.)

$$\frac{+16a^3b^3c^2d}{+4ab^2cd} = +4a^2bc$$

(2.)

$$\frac{-18a^2b^2x^5y^4z}{-6abx^3y^2z} = 3abx^2y^2.$$

(3.)

$$\frac{+15a^3x^2y^4z^3}{-5a^2xy^3z^2} = -3axyz$$

(4.)

$$\frac{-21p^3y^3x^2z}{+7p^3yx} = -3y^2xz.$$

5. Divide $21fg^2h$ by $7fg$. *Ans.* $3gh$.
6. Divide $84a^2x^3y$ by $-12axy$. *Ans.* $-7ax^2$.
7. Divide $-36a^2b^3c$ by $18ab$. *Ans.* $-2ab^2c$.
8. Divide $-25a^2x^2y^4$ by $-5x^2y^2$. *Ans.* $5a^2y^2$.
9. Divide $72a^4b^4c^2$ by $-36a^2b^2$. *Ans.* $-2a^2b^2c^2$.
10. Divide $-14a^2x^3y^4$ by $-7ax^2y^3$. *Ans.* $2axy$.
11. Divide $14ax^3z$ by $7axz$. *Ans.* $2x^2$.
12. Divide $-24fg^2h$ by $6g$. *Ans.* $-4fgh$.
13. Divide $a^6b^6c^6$ by a^3b^3c and that result by ab^2c^3. *Ans.* a^2bc^2.

14. Divide $300a^3c^3$ by $10ac$ and that result by $6a^2c$. *Ans.* $5c$.

15. Divide $-25a^3b^3c^2$ by $-5a^2bc^2$. *Ans.* $5ab^2$.

16. Divide $-36ax^3y^5$ by $-9ax^2y$. *Ans.* $4xy^4$.

Explanation of the Exponent 0.

31. In dividing one monomial by another it often happens that the exponents of a letter are the same in both dividend and divisor, in which case that letter disappears from the quotient. It may however be retained with the exponent 0. Thus, by the rule, we have,

$$\frac{a^m}{a^m} = a^{m-m} = a^0; \text{ but } \frac{a^m}{a^m} = 1; \ \therefore \ a^0 = 1.$$

From this we infer, that, *the* 0 *power of any quantity is equal to* 1, *and that* 1 *is equal to the* 0 *power of any quantity. Any quantity may therefore be introduced into a term, as a factor, by giving it the exponent* 0.

Explanation of Negative Exponents.

32. If the exponent of any letter in the dividend is less than the exponent of the same letter in the divisor, the exponent of that letter in the quotient will be negative. Thus, by the rule, we have,

$$\frac{a^2}{a^5} = a^{-3}; \text{ but } \frac{a^2}{a^5} = \frac{1}{a^3}; \ \therefore \ a^{-3} = \frac{1}{a^3}.$$

Hence, we infer that *a quantity with a negative ex-*

ponent is equivalent to the reciprocal of the same quantity with an equal positive exponent.

We also infer that *a factor may be transferred from the denominator to the numerator of a fraction, or the reverse, by changing the sign of its exponent.*

These conclusions are in accordance with the principle explained in Art. 6; for if a positive exponent indicates that a quantity is to be taken a certain number of times as a factor, an equal negative exponent should indicate that the quantity is to be taken the same number of times as a divisor.

It will be shown hereafter that quantities having negative exponents can be operated on by the rules that are given for operating on quantities with positive exponents. This principle often enables us to change an indicated quotient to a simpler form without altering its value. Such a change is called *reduction.*

EXAMPLES.

(1.)

$$\frac{7a^3b^2c}{4a^2b^2} = \frac{7}{4}ab^0c = \frac{7}{4}ac.$$

(2.)

$$\frac{6ab^3c}{3a^2b^4c^2} = 2a^{-1}b^{-1}c^{-1} = \frac{2}{abc}.$$

3. Reduce $\frac{15ab^2c}{5a^2b^3}$. *Ans.* $3a^{-1}b^{-1}c$, or $\frac{3c}{ab}$.

4. Reduce $\frac{17xyz}{5x^3y^2z}$. *Ans.* $\frac{17}{5}x^{-2}y^{-1}$, or $\frac{17}{5x^2y}$.

5. Reduce $\frac{24fgx}{6fg^2x^3}$. *Ans.* $4f^0g^{-1}x^{-2}$, or $\frac{4}{gx^2}$.

6. Reduce $\frac{-8a^{-1}x^{-2}}{4ax^2}$. *Ans.* $-2a^{-2}x^{-4}$, or $-\frac{2}{a^2x^4}$.

7. Reduce $\frac{-6a^{-3}x^{-4}}{-3a^{-4}x^{-5}}$. *Ans.* $2ax$.

8. Reduce $72a^2b^3 \div 12a^3$. *Ans.* $6a^{-1}b^3$, or $\frac{6b^3}{a}$.

9. Reduce $\left(\frac{-36ax^4}{12a^2x}\right) \div 3ax^5$. *Ans.* $-\frac{1}{a^2x^2}$.

10. Reduce $(14ax \div 7yz) \div 2xz$. *Ans.* $\frac{a}{yz^2}$.

Division of Polynomials by Monomials.

33. By reversing the rule for multiplying a polynomial by a monomial, we have the following rule for dividing a polynomial by a monomial:

RULE.

Divide each term of the polynomial by the monomial, and connect the quotients by their proper signs.

EXAMPLES.

(1.)

$$\frac{6ab-8ax+4a^2y}{2a}=3b-4x+2ay.$$

(2.)

$$\frac{10a^2x-15x^2}{5x}=2a^2-3x.$$

3. Divide $5xy + 20x^2y - 45axy$ by $5xy$.
Ans. $1 + 4x - 9a$.

4. Divide $-9a^2bc - 12ab^2c + 15abc^2$ by $-3abc$.
Ans. $3a + 4b - 5c$.

5. Divide $14a^2xy^3 - 7aby^4 - 13y^4$ by $7a^3x^3y^3$.
Ans. $2a^{-1}x^{-2} - a^{-2}bx^{-3}y - \frac{13}{7}a^{-3}x^{-3}y$.

6. Divide $27a^m - 18a^{2m}b^n - 21a^p$ by $3a^m$.
Ans. $9 - 6a^mb^n - 7a^{p-m}$.

7. Divide $12a^4(a + x)^2 - 18a^3(a + x)^3 + 24a^2(a + x)^4$ by $6a^2(a + x)^2$. *Ans.* $2a^2 - 3a(a + x) + 4(a + x)^2$.

Division of Polynomials.

34. To deduce a rule for dividing one polynomial by another, take the following example:

$$\begin{array}{ll}
\textit{Dividend.} & \textit{Divisor.} \\
9x^3 + 12x^2 + 16x + 8 & \left| \; 3x + 2 \right. \\ \cline{2-2}
\underline{9x^3 + 6x^2 } & \left| \; 3x^2 + 2x + 4; \right. \quad \textit{Quotient.} \\
6x^2 + 16x & \\
\underline{6x^2 + 4x } & \\
12x + 8 & \\
\underline{12x + 8} & \\
0 & \textit{Remainder.}
\end{array}$$

The dividend and divisor are both arranged with reference to the same letter; the quotient of the first term of the dividend, by the first term of the divisor, is therefore the first term of the quotient. The product of the divisor by this term, subtracted from the dividend, gives a new dividend, which is treated in the same way, and so on to the end of the process.

For convenience of multiplication, the divisor is written on the right of the dividend, and the quotient is written under the divisor. In all other respects, the operation is entirely similar to division in Arithmetic.

Since all similar cases may be treated in the same manner, we have the following rule for the division of polynomials:

RULE.

I. Arrange both polynomials with reference to the same letter.

II. Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the divisor by this term, and subtract the product from the dividend.

III. Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient. Multiply the divisor by this term, and subtract the product from the first remainder, and so on.

IV. Continue the operation, until a remainder is found equal to 0, or one whose first term is not divisible by that of the divisor.

If a remainder is found equal to 0, the division is exact. If a remainder is found whose first term is not divisible by that of the divisor, without giving rise to fractions, the exact division is impossible. In that case, write the last remainder over the divisor and add the result to the quotient already found.

EXAMPLES.

(1.)

$$\begin{array}{rrr|l}
6a^3x^3 & +\ 13a^2x^2 & +\ 6ax & 2a^2x^2 + 3ax \\ \cline{4-4}
6a^3x^3 & +\ 9a^2x^2 & & 3ax \quad + 2 \\ \cline{1-2}
 & 4a^2x^2 & +\ 6ax & \\
 & 4a^2x^2 & +\ 6ax & \\ \cline{2-3}
 & & 0 &
\end{array}$$

(2.)

$$\begin{array}{rrrrr|l}
a^4 & +\ a^3x & +\ a^2x & +\ ax^2 & -\ 2x & a + x \\ \cline{6-6}
a^4 & +\ a^3x & & & & a^3 + ax - \dfrac{2x}{a+x} \\ \cline{1-4}
 & & +\ a^2x & +\ ax^2 & & \\
 & & +\ a^2x & +\ ax^2 & & \\ \cline{3-5}
 & & & & -\ 2x &
\end{array}$$

Here the quotient is fractional, and the division is not exact.

(3.)

$$\begin{array}{l|l}
1+x & 1-x \\
\cline{2-2}
1-x & 1+2x+2x^2+2x^3+, \text{etc.} \\
\cline{1-1}
\quad +2x & \\
\quad +2x-2x^2 & \\
\cline{1-1}
\qquad\qquad +2x^2 & \\
\qquad\qquad +2x^2-2x^3 & \\
\cline{1-1}
\qquad\qquad\qquad\quad +2x^3 &
\end{array}$$

In this example, the operation does not terminate, but may be continued to any desired extent.

EXAMPLES.

1. Divide $a^2+4ax+4x^2$ by $a+2x$.
Ans. $a+2x$.

2. Divide $a^3-3a^2x+3ax^2-x^3$ by $a-x$.
Ans. $a^2-2ax+x^2$.

3. Divide $a^3+5a^2x+5ax^2+x^3$ by $a+x$.
Ans. $a^2+4ax+x^2$.

4. Divide $a^4-4a^3y+6a^2y^2-4ay^3+y^4$ by $a^2-2ay+y^2$
Ans. $a^2-2ay+y^2$.

5. Divide a^4-b^4 by $a^3+a^2b+ab^2+b^3$.
Ans. $a-b$.

6. Divide $12x^4-192$ by $3x-6$.
Ans. $4x^3+8x^2+16x+32$.

7. Divide $x^6-3x^4y^2+3x^2y^4-y^6$ by $x^3-3x^2y+3xy^2-y^3$.
Ans. $x^3+3x^2y+3xy^2+y^3$.

8. Divide $x^{4n}+x^{2n}y^{2n}+y^{4n}$ by $x^{2n}+x^ny^n+y^{2n}$
Ans. $x^{2n}-x^ny^n+y^{2n}$.

9. Divide $a^2 - b^2 + 2bc - c^2$ by $a - b + c$.
Ans. $a + b - c$.

10. Divide $x^4 - 6x^2y^2 - 16xy^3 - 15y^4$ by $x^2 + 2xy + 3y^2$.
Ans. $x^2 - 2xy - 5y^2$.

11. Divide $ax^3 - a^2x^2 - bx^2 + b^2$ by $ax - b$.
Ans. $x^2 - ax - b$.

12. Divide $mpx^3 + mqx^2 - npx^2 - mrx - nqx + nr$ by $mx - n$.
Ans. $px^2 + qx - r$.

13. Divide $a^3x^3 - a^3x + a^2x^2 + 2a^2x - 2a^2 + 2ax + ax^2 - ax^3 - x^3$ by $a^2x + 2a - x^2$.
Ans. $ax - a + x$.

14. Divide $-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8$ by $2a^{-3}x^3 - 3ax^4$.
Ans. $-a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$.

15. Divide $a^3 - 3a^2x + x^3$ by $a + x$.
Ans. $a^2 - 4ax + 4x^2 - \dfrac{3x^3}{a+x}$.

16. Divide $a^5 + a^3b^2 + 2a^2b^3 - b^5$ by $a^2 - ab + b^2$.
Ans. $a^3 + a^2b + ab^2 + 2b^3 + \dfrac{ab^4 - 3b^5}{a^2 - ab + b^2}$.

17. Divide $x^3 + ax^2 + bx + c$ by $x - r$.
Ans. $x^2 + rx + ax + r^2 + ar + b + \dfrac{r^3 + ar^2 + br + c}{x - r}$.

18. Divide $1 + 2x$ by $1 - 3x$.
Ans. $1 + 5x + 15x^2 + 45x^3 +$ &c.

19. Divide $1 + 2x$ by $1 - x - x^2$.
Ans. $1 + 3x + 4x^2 + 7x^3 +$ &c.

20. Divide 1 by $1 + x$.
Ans. $1 - x + x^2 - x^3 + x^4 +$ &c.

V. USEFUL FORMULAS.

Definitions.

35. A **formula**, is an algebraic expression of a general rule, or principle.

Formulas are used to shorten algebraic operations, such as the formation of powers, factoring, and the like.

Illustration.

36. Let the following formulas be verified by actual multiplication:

1°. $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2.$

2°. $(x-y)^2 = (x-y)(x-y) = x^2 - 2xy + y^2.$

3°. $(x+y)(x-y) = x^2 - y^2.$

If we suppose x and y to represent any two quantities, and then translate these formulas into words, we have the following principles:

1°. *The square of the sum of any two quantities, is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

2°. *The square of the difference of any two quantities, is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

3°. *The product of the sum and difference of any two quantities, is equal to the square of the first, minus the square of the second.*

The method of applying these principles is shown in the following

EXAMPLES.

1. Let it be required to find the square of $2a + 3x$.

The square of $2a$, is $4a^2$; twice the product of $2a$ and $3x$, is $12ax$; the square of $3x$, is $9x^2$.

Hence, by the first principle,

$$(2a + 3x)^2 = 4a^2 + 12ax + 9x^2.$$

2. Find the square of $2a - 3x$. By the second principle, we have, as before,

$$(2a - 3x)^2 = 4a^2 - 12ax + 9x^2.$$

3. Find the product of $2a + 3x$, and $2a - 3x$. By the third principle, we have, as before,

$$(2a + 3x)(2a - 3x) = 4a^2 - 9x^2.$$

In like manner let the following operations be performed:

4. Find the square of $ax + by$.
Ans. $a^2x^2 + 2abxy + b^2y^2$.

5. Find the square of $7x^2 + 3y^2$.
Ans. $49x^4 + 42x^2y^2 + 9y^4$.

6. Find the square of $8ab + 4cd$.
Ans. $64a^2b^2 + 64abcd + 16c^2d^2$.

7. Find the square of $2ac - 3d$.
Ans. $4a^2c^2 - 12acd + 9d^2$.

8. Find the square of $16xy - 7y^2$.
Ans. $256x^2y^2 - 224xy^3 + 49y^4$.

9. Find the square of $2ab - cd$.

Ans. $4a^2b^2 - 4abcd + c^2d^2$.

10. Find the product of $2a + 3x$, and $2a - 3x$.

Ans. $4a^2 - 9x^2$.

11. Find the product of $7b + 4c$, and $7b - 4c$.

Ans. $49b^2 - 16c^2$.

12. Find the product of $8xy + 3x^2$, and $8xy - 3x^2$.

Ans. $64x^2y^2 - 9x^4$.

By reversing these operations, the squares and products above found may be resolved into binomial factors.

The following additional formulas may be verified by actual multiplication, or division, with the exception of the ninth and tenth. The demonstration of these will be given in the appendix.

4°. $(x + a)(x + b) = x^2 + (a + b)x + ab$.

5°. $(x^2 + xy + y^2)(x - y) = x^3 - y^3$.

6°. $(x^2 - xy + y^2)(x + y) = x^3 + y^3$.

7°. $(x^2 + xy + y^2)(x^2 - xy + y^2) = x^4 + x^2y^2 + y^4$.

8°. $(x + y)(x - y)(x^2 + y^2) = x^4 - y^4$.

9°. $\dfrac{x^n - y^n}{x - y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \text{\&c.} + y$

10°. $\dfrac{x^{mn} - y^{mn}}{x^n - y^n} = x^{mn-n} + x^{mn-2n}y^n + x^{mn-3n}y^{2n} + \text{\&c.}$
$+ y^{mn-n}$

VI. FACTORING.

Definitions.

37. **Factoring** is the operation of separating, or resolving, a quantity into factors.

No general rule can be given for factoring: in most cases the operation is performed by inspection and trial. The methods of proceeding are best illustrated by examples.

Methods of Factoring.

38. If every term of a polynomial contains the same monomial factor, that factor is one factor of the polynomial, and the other factor is equal to the quotient of the polynomial by the monomial factor.

EXAMPLES.

1. Factor the polynomial $8a^2x^2 + 4a^3x$.

Here, we see that $4a^2x$ is a factor common to each term, hence it is one of the required factors. Dividing by $4a^2x$, we have the quotient, $2x + a$, which is the other factor; or,

$$8a^2x^2 + 4a^3x = 4a^2x(2x + a).$$

In like manner, the following polynomials may be factored.

2. Factor $7a^2bc^3 - 28abc$. *Ans.* $7abc(ac^2 - 4)$.

3. Factor $4x^4y^2 - 2x^2y^2$. *Ans.* $2x^2y^2(2x^2 - 1)$.

4. Factor $6x^2y^2 + 12xy^3$. *Ans.* $6xy^2(x + 2y)$.

5. Factor $2a^2b + abc - abd$. *Ans.* $ab(2a + c - d)$.

6. Factor $7x^3y^2 - 7x^2y^3 + 7x^2y^2z$.
Ans. $7x^2y^2(x - y + z)$.

7. Factor $15a^2cd + 20ac^2d - 15acd^2$.
Ans. $5acd(3a + 4c - 3d)$.

39. If two terms of a trinomial are squares, and the third term is equal to twice the product of their square roots, the trinomial may be factored by means of principle 1°, Art. 35.

8. Factor $a^2 + 2ab + b^2$. *Ans.* $(a + b)(a + b)$.

9. Factor $4x^2 + 12xy + 9y^2$.
Ans. $(2x + 3y)(2x + 3y)$.

10. Factor $x^2 + 12x + 36$. *Ans.* $(x + 6)(x + 6)$.

11. Factor $4x^4 + 4x^2y + y^2$.
Ans. $(2x^2 + y)(2x^2 + y)$.

12. Factor $4a^2b^2 + 12abc + 9c^2$
Ans. $(2ab + 3c)(2ab + 3c)$.

13. Factor $16a^4y^4 + 8a^2y^4z^2 + y^4z^4$.
Ans. $(4a^2y^2 + y^2z^2)(4a^2y^2 + y^2z^2)$.

40. If two terms of a trinomial are squares, and the third term is equal to *minus* twice the product of their square roots, the trinomial may be factored by means of principle 2°, Art. 35.

14. Factor $a^2 - 2ab + b^2$. *Ans.* $(a - b)(a - b)$.

15. Factor $a^2x^2 - 2acx + c^2$. *Ans.* $(ax - c)(ax - c)$.

16. Factor $4x^2 - 4xy + y^2$. *Ans.* $(2x - y)(2x - y)$.

17. Factor $9a^2b^2 - 24a^2bc + 16a^2c^2$.
Ans. $(3ab - 4ac)(3ab - 4ac)$.

18. Factor $4x^4 - 4x^2y + y^2$. *Ans.* $(2x^2 - y)(2x^2 - y)$.

19. Factor $36x^2 - 24xy + 4y^2$.
Ans. $(6x - 2y)(6x - 2y)$.

20. Factor $4x^2y^2 - 4xyz + z^2$.
Ans. $(2xy - z)(2xy - z)$.

41. If the two terms of a binomial are squares, and have contrary signs, the binomial may be factored by means of principle 3°, Art. 35.

21. Factor $a^2 - b^2$. *Ans.* $(a + b)(a - b)$.

22. Factor $4x^2 - 9y^2$. *Ans.* $(2x + 3y)(2x - 3y)$.

23. Factor $a^2c^2 - b^2d^2$. *Ans.* $(ac + bd)(ac - bd)$.

24. Factor $9a^2x^2 - 16a^2y^2$.
Ans. $(3ax + 4ay)(3ax - 4ay)$.

25. Factor $25a^4b^4x^4 - 4z^2$.
Ans. $(5a^2b^2x^2 + 2z)(5a^2b^2x^2 - 2z)$.

26. Factor $49x^4 - 16y^2$. *Ans.* $(7x^2+4y)(7x^2-4y)$.

The following examples may be factored by means of formula 4°, Art. 35:

27. $x^2 + 13x + 42 = x^2 + (6 + 7)x + 6 \times 7$
$= (x + 6)(x + 7)$.

28. $x^2 + 2x - 15 = x^2 + (5 - 3)x - 3 \times 5$
$= (x - 3)(x + 5)$.

29. $x^2 - 15x + 56 = x^2 - (7 + 8)x - 7 \times - 8$
$= (x - 7)(x - 8)$.

30. $x^2 - x - 72 = x^2 + (8 - 9)x - 9 \times 8$
$= (x + 8)(x - 9)$.

The following examples may be factored by means of formulas 5°, 6°, 7°, and 8°, Art. 35:

31. $8a^3 - b^3 = (2a - b)(4a^2 + 2ab + b^2)$.

32. $a^3 + 64m^3 = (a + 4m)(a^2 - 4am + 16m^2)$.

33. $16a^4 + 36a^2b^2 + 81b^4$
$= (4a^2 + 6ab + 9b^2)(4a^2 - 6ab + 9b^2)$.

34. $a^4b^4 - 81c^4 = (a^2b^2 + 9c^2)(a^2b^2 - 9c^2)$
$= (a^2b^2 + 9c^2)(ab + 3c)(ab - 3c).$

Let the following miscellaneous examples be factored:

35. $9x^4y^2 + 24x^3y^3 + 16x^2y^4.$
Ans. $(3x^2y + 4xy^2)(3x^2y + 4xy^2).$

36. $4x^2 - 12xy + 9y^2.$ *Ans.* $(2x - 3y)(2x - 3y).$

37. $a^2b^2c^2 - c^2d^2.$ *Ans.* $c^2(ab + d)(ab - d).$

38. $x^2 + 9x + 18.$ *Ans.* $(x + 6)(x + 3).$

39. $2a^2x^2 - 2b^2x^2.$ *Ans.* $2x^2(a + b)(a - b).$

40. $a^2 - b^2 + 2bc - c^2.$
Ans. $a^2 - (b - c)^2 = (a + b - c)(a - b + c).$

41. $a^4 - 9a^2b^2 - 6abc^2 - c^4.$
Ans. $a^4 - (3ab + c^2)^2 = (a^2 + 3ab + c^2)(a^2 - 3ab - c^2).$

CHAPTER III.

GREATEST COMMON DIVISOR, AND LEAST COMMON MULTIPLE.

I. GREATEST COMMON DIVISOR.

Definitions.

42. A **common divisor** of two quantities, is a quantity that will divide both without a remainder. Thus, $3a^2b$, is a common divisor of $9a^2b^2c$ and $3a^2b^2-6a^3b^3$.

43. A **simple or prime factor** is one that cannot be resolved into any other factors.

Every prime factor, common to two quantities, is a common divisor of those quantities. The continued product of any number of prime factors, common to two quantities, is also a common divisor of those quantities.

44. The **greatest common divisor** of two quantities, is the continued product of all the prime factors that are common to both.

It is called the greatest common divisor, because it is greater with respect to its coefficients and exponents than any other common divisor.

There are two methods of finding the greatest common divisor: *by factoring*, and *by continued division.*

Method by Factoring.

45. If both quantities can be resolved into prime factors by the methods already given, the greatest common divisor may be found by the following

RULE.

I. *Resolve both quantities into prime factors.*

II. *Find the continued product of all the prime factors common to both; it will be the greatest common divisor required.*

EXAMPLES.

1. Let it be required to find the greatest common divisor of $42abx$ and $70acx$:

Factoring, we have,

$$\begin{aligned} 42abx &= 7a \times 2x \times 3b, \\ 70acx &= 7a \times 2x \times 5c. \end{aligned}$$

The factors $7a$ and $2x$ are common: hence, the greatest common divisor is $7a \times 2x$, or $14ax$.

2. Find the greatest common divisor of $3ax^2 + 3x^3$ and $2ay + 2xy$:

Factoring, we have,

$$\begin{aligned} 3ax^2 + 3x^3 &= 3x^2(a+x), \\ 2ay + 2xy &= 2y\,(a+x); \end{aligned}$$

hence, the greatest common divisor is $a + x$.

3. Find the greatest common divisor of $2a^3 - 4a^2b + 2ab^2$ and $2a^3 - 2ab^2$:

Factoring, we have,

$$\begin{aligned} 2a^3 - 4a^2b + 2ab^2 &= 2a(a^2 - 2ab + b^2) &&= 2a(a - b)\ (a - b), \\ 2a^3 - 2ab^2 &= 2a(a^2 - b^2) &&= 2a(a - b)\ (a + b); \end{aligned}$$

hence, the greatest common divisor is $2a(a - b)$, or $2a^2 - 2ab$.

4. Find the greatest common divisor of $56acdx^2y$ and $24afx^2y$. *Ans.* $8ax^2y$.

5. Find the greatest common divisor of $4a^2c - 4acx$ and $3a^2g - 3agx$. *Ans.* $a(a - x)$, or $a^2 - ax$.

6. Find the greatest common divisor of $x^3 - y^3$ and $x^2 - y^2$. *Ans.* $x - y$.

7. Find the greatest common divisor of $4c^2 - 9x^2$ and $4c^2 - 12cx + 9x^2$. *Ans.* $2c - 3x$.

8. Find the greatest common divisor of $4ax^3 - 4axy^2$ and $12a^2x^2 - 12a^2y^2$. *Ans.* $4ax^2 - 4ay^2$.

9. Find the greatest common divisor of $2a^3x + 4a^2bx + 2ab^2x$ and $4a^2x^3 + 8abx^3 + 4b^2x^3$.

Ans. $2a^2x + 4abx + 2b^2x$.

The principles just explained, together with those given in arithmetic, (Complete Arithmetic, Arts. 53–56), enable us to find the greatest common divisor, when the quantities considered are monomials.

Method by Continued Division.

46. The general method of finding the greatest common divisor of any two polynomials depends on the following principle:

1°. *If the first of two given polynomials is divided* *by the second and a remainder found, the greatest common divisor of this remainder and the second polynomial is the greatest common divisor of the given polynomials.*

To demonstrate this principle let M and N be two polynomials, whose greatest common divisor is D; also let M be divided by N, and denote the quotient by Q and the remainder by R. Then, from the nature of division, we shall have,

$$M = NQ + R.$$

Now, any quantity that will divide M will divide its equal NQ + R, and the reverse; also, any quantity that will divide N will divide Q times N, or NQ: hence, any quantity that will divide M and N will divide R, that is, R contains all the factors that are common to M and N; in like manner, any quantity that will divide R and N will divide NQ + R, and consequently its equal M, that is, M contains all the factors that are common to R and N: hence, the greatest common divisor of R and N is also the greatest common divisor of M and N, *which was to be shown.*

In applying the principle just explained, the operations may be simplified by means of the following additional principles:

2°. *The monomial factors common to each polynomial may be suppressed; if, however, any factor so suppressed is common to the two, it must be set aside as a factor of the greatest common divisor.*

3°. *Either polynomial may be multiplied by any factor that is not contained in the other.*

From the principles given above, we deduce the following rule for finding the greatest common divisor of two polynomials:

RULE.

I. Suppress all the monomial factors of each polynomial; if any factor suppressed is common to the two, set it aside as a factor of the common divisor.

II. Multiply the first polynomial by the simplest factor that will make its first term divisible by the first term of the second polynomial; then divide this result by the second polynomial, continuing the division as far as possible.

III. Take the second polynomial as a dividend, and the first remainder as a divisor, and proceed as before; and so on, till a remainder is found that will divide the preceding divisor. This remainder, multiplied by the factors set aside, will give the greatest common divisor.

In dividing the first polynomial by the second, any partial remainder may be multiplied by such a factor as will make its first term divisible by the first term of the second polynomial.

EXAMPLES.

1. Let it be required to find the greatest common divisor of $16a^2x^2 + 48a^2x + 36a^2$, and $12ax^2 + 10ax - 12a$.

Suppressing the factor $4a^2$ in the first, and $2a$ in the second, and setting aside the factor $2a$, which is common to both, we have the polynomials,

$$4x^2 + 12x + 9, \quad \text{and} \quad 6x^2 + 5x - 6;$$

multiplying the first by 3, and proceeding according to the rule, we have the

FIRST OPERATION.

$$\begin{array}{r|l} 12x^2 + 36x + 27 & \underline{6x^2 + 5x - 6} \\ \underline{12x^2 + 10x - 12} & 2 \\ 26x + 39\,, & \qquad \textit{first remainder.} \end{array}$$

Suppressing the factor 13 in the first remainder and proceeding as before, we have the

SECOND OPERATION.

$$\begin{array}{r|l} 6x^2 + 5x - 6 & \underline{2x + 3} \\ \underline{6x^2 + 9x \phantom{{}-6}} & 3x - 2 \\ -4x - 6 & \\ \underline{-4x - 6} & \\ 0\,, & \qquad \textit{second remainder.} \end{array}$$

Hence, the greatest common divisor is $2a(2x + 3)$.

2. Find the greatest common divisor of x^3-5x^2+7x-3 and $x^2 + x - 12$.

FIRST OPERATION.

$$\begin{array}{r|l} x^3 - 5x^2 + 7x - 3 & \underline{x^2 + x - 12} \\ \underline{x^3 + x^2 - 12x \phantom{{}-3}} & x - 6 \\ -6x^2 + 19x - 3 & \\ \underline{-6x^2 - 6x + 72} & \\ 25x - 75\,, & \qquad \textit{first remainder.} \end{array}$$

Suppressing the factor 25 in the first remainder, we have the following

OPERATION.

$$\begin{array}{r|l} x^2 + x - 12 & x - 3 \\ \underline{x^2 - 3x\phantom{{}-12}} & \overline{x + 4} \\ 4x - 12 & \\ \underline{4x - 12} & \\ 0\;, & \quad \textit{second remainder} \end{array}$$

Hence, $x - 3$ is the required common divisor.

3. Find the greatest common divisor of $3a^2x^3 - 3a^2y^3$ and $6ax^2 - 6ay^2$. *Ans.* $3a(x - y)$.

Find the greatest common divisor of the polynomials in each of the following examples:

4. Of $a^2 - 4$ and $a^2 + 4a + 4$. *Ans.* $a + 2$.

5. Of $a^3 - ab^2$ and $a^2 + 2ab + b^2$. *Ans.* $a + b$.

6. Of $x^5 - x^3b^2$ and $x^4 - b^4$. *Ans.* $x^2 - b^2$.

7. Of $x^2 + 2x - 3$ and $x^2 + 5x + 6$. *Ans.* $x + 3$.

8. Of $3x^2y + 3xy^2$ and $3x^2 + 6xy + 3y^2$. *Ans.* $3x + 3y$.

9. Of $x^4 + ax^3 - a^3x - a^4$ and $x^4 + a^2x^2 + a^4$. *Ans.* $x^2 + ax + a^2$.

10. Of $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$. *Ans.* $5x^2 - 1$.

Application to three or more Quantities.

47. To find the greatest common divisor of three, or more quantities, we first find that of the first and second quantities; then that of the result and the third quantity; and so on to the last: the final result is the required divisor.

EXAMPLES.

1. Find the greatest common divisor of $2a^2x^2$, $4x^2y^2$, and $8x^3y$.

The greatest common divisor of $2a^2x^2$ and $4x^2y^2$, is $2x^2$, which exactly divides $8x^3y$; hence, $2x^2$ is the divisor required.

2. Find the greatest common divisor of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

The greatest common divisor of the first and second, is $x + 4$, and that of this result and the third polynomial, is $x + 4$, which is the divisor sought.

Find the greatest common divisor of each of the following groups of quantities:

3. Of $3a^nx^{n-1}$, $6a^{2n}x^{n+1}$, and $21a^{n-1}x^{2n}$.
Ans. $3a^{n-1}x^{n-1}$.

4. Of $4ax^2y$, $16abx^2$, and $24acx^2$. *Ans.* $4ax^2$.

5. Of $7a^2 + 7ab$, $4ab + 4b^2$, and $2ac + 2bc$.
Ans. $a + b$.

6. Of $3x^2 - 6x$, $2x^3 - 4x^2$, and $x^2y - 2xy$.
Ans. $x^2 - 2x$.

7. Of $3x^2 + 6xy$, $2xy + 4y^2$, and $4xz + 8zy$.
Ans. $x + 2y$.

8. Of $3a^2 - 3b^2$, $3a^2 + 6ab + 3b^2$, and $3axy + 3bxy$.
Ans. $3(a + b)$.

9. Of $x^2 - 9$, $x^2 - 3x - 18$, and $x^2 + 11x + 24$.
Ans. $x + 3$.

10. Of $x^2-3x-28$, $x^2-11x+28$, and $x^2-15x+56$.
Ans. $x - 7$.

11. Of x^4-5x^2+6, x^4-7x^2+12, and x^4+2x^3-15. *Ans.* x^2-3.

12. Of x^3+5x^2+7x+3, x^3+3x^2-x-3, and x^3+x^2-5x+3. *Ans.* $x+3$.

II. LEAST COMMON MULTIPLE.

Definitions.

48. One quantity is a **multiple** of another, when the former can be divided by the latter without a remainder. Thus, $8a^2b$, is a multiple of 8, also of a^2, and of b.

49. A quantity is a **common multiple** of two or more quantities, when it can be divided by each, separately, without a remainder. Thus, $24a^3x^2$, is a common multiple of $6ax$ and $4a^2x$.

50. The **least common multiple** of two or more quantities, is the simplest quantity that can be divided by each, without a remainder. Thus, $12a^2b^2x^2$ is the least common multiple of $2a^2x$, $4ab^2$, and $6a^2b^2x^2$.

There are two methods of finding the least common multiple of two or more quantities: the *method by factoring*, and *the method by means of the greatest common divisor.*

Method by Factoring.

51. To deduce a rule for finding the least common multiple by the method by factoring, let it be required to find the least common multiple of $12a^2x$, $27ax^3$, and

$30a^3xy$. Resolving these quantities into their prime factors, we have,

$$2\times 2\times 3aax, \quad 3\times 3\times 3axxx, \quad \text{and} \quad 2\times 3\times 5aaaxy.$$

In order that the required multiple may be divisible by the first of the given quantities, it must contain the factor 2 *twice;* in order that it may be divisible by the second quantity, it must contain each of the factors 3 and x *three times;* and in order that it may be divisible by the third quantity, it must contain the factor a, *three times* and each of the factors 5 and y, *once:* it is therefore equal to

$$2\times 2\times 3\times 3\times 3\times 5aaaxxxy, \quad \text{or} \quad 540a^3x^3y$$

Since all similar cases may be treated in the same manner, we have the following

RULE.

I. Resolve all of the quantities into their prime factors.

II. Take each factor the greatest number of times it enters any of the quantities, and form the continued product of these factors; it will be the common multiple required.

EXAMPLES.

1. Find the least common multiple of $6ab^2$ and $18a^2b$,

Factoring, we have,

$$6ab^2 = 2\times 3abb, \quad \text{and} \quad 18a^2b = 2\times 3\times 3aab.$$

Hence, the required multiple is

$$2\times 3\times 3aabb = 18a^2b^2.$$

2. Find the least common multiple of $6a^2xy$, $8ax^2$, and $12x^2y^2$.

Factoring, we have,

$$\begin{aligned} 6a^2xy &= 2 \,.\, 3aaxy, \\ 8ax^2 &= 2 \,.\, 2 \,.\, 2axx, \\ 12x^2y^2 &= 2 \,.\, 2 \,.\, 3xxyy \end{aligned}$$

hence, the least common multiple is,

$$2 \,.\, 2 \,.\, 2 \,.\, 3 \,.\, aaxxyy. \quad \text{or} \quad 24a^2x^2y^2.$$

3. Find the least common multiple of $a^2 - b^2$ and $a^2 - 2ab + b^2$.

Factoring, we have,

$$\begin{aligned} a^2 - b^2 &= (a + b)\,(a - b) \\ a^2 - 2ab + b^2 &= (a - b)\,(a - b); \end{aligned}$$

hence, the least common multiple is

$$(a - b)\,(a - b)\,(a + b), \quad \text{or} \quad a^3 - ab^2 - a^2b + b^3.$$

Find the least common multiple of each of the following groups of quantities:

4. Of $15x^2y^2$ and $6x^3y$. *Ans.* $30x^3y^2$.

5. Of $3x^2yz$, $6xy^3$, and $9xyz$. *Ans.* $18x^2y^3z$.

6. Of $3ab^2$, $6ac^3$, $4c^3d$, and b^2c^2. *Ans.* $12ab^2c^3d$.

7. Of $ax - bx$, $ay - by$, and x^2y^2.
Ans. $ax^2y^2 - bx^2y^2$.

8. Of $a - b$, $a^2 - b^2$, and $a^2 - 2ab + b^2$
Ans. $(a - b)^2\,(a + b)$.

9. Of $8x^2(x - y)$, $15x^5(x - y)^2$, and $12x^3(x^2 - y^2)$.
Ans. $120x^5(x^2 - y^2)\,(x - y)$.

10. Of $2a^2(a + x)$ and $4ax(a - x)$.
Ans. $4a^2x(a^2 - x^2)$.

Method by Means of the Greatest Common Divisor.

52. If the quantities cannot be factored conveniently by methods already given, their least common multiple may be found by means of the greatest common divisor. This method depends on the following principle:

The least common multiple of two quantities is equal to one of the quantities multiplied by the quotient, obtained by dividing the other quantity by their greatest common divisor.

To demonstrate this principle let P and P′ denote any two quantities, and let D be their greatest common divisor; let Q and Q′ be the quotients of P and P′ by D. Then, we have,

$$P = Q \times D, \quad \text{and} \quad P' = Q' \times D.$$

Since Q and Q′ have no common factor, their least common multiple is $Q \times Q'$; consequently the least common multiple of P and P′, is $D \times Q \times Q'$; or, since $P = DQ$, we have the least common multiple equal to $P \times Q'$, *which was to be shown.*

From the preceding principle we have the following rule for finding the least common multiple of two quantities:

RULE.

Find their greatest common divisor; divide one of them by it, and multiply the other by the quotient.

EXAMPLES.

1. Find the least common multiple of $a^3 - x^3$ and $a^2 - x^2$.

Their greatest common divisor is $a - x$. Hence, their least common multiple is,

$$\frac{a^2 - x^2}{a - x} \times (a^3 - x^3), \quad \text{or} \quad a^4 + a^3x - ax^3 - x^4.$$

2. Find the least common multiple of $2x - 1$ and $4x^2 - 1$.

The greatest common divisor of the two, is $2x - 1$: hence, the required multiple is,

$$\frac{2x - 1}{2x - 1} \times (4x^2 - 1) = 4x^2 - 1.$$

3. Find the least common multiple of $x^2 + 7x + 12$, and $x^2 + 8x + 15$. *Ans.* $(x + 3)(x + 4)(x + 5)$.

If there are more than two quantities, we find the least common multiple of the first and second, then that of this result and the third, and so on, to the last.

Find the least common multiples of each of the following groups of quantities:

4. Of $8a^2$, $12a^3$, and $20a^4$. *Ans.* $120a^4$.

5. Of $x^2 + 5x + 6$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.
Ans. $(x^2 + 2x - 8)(x^2 + 5x + 6)$.

6. Of $x - 1$, $x^2 - 1$, and $x^2 + 4x - 5$.
Ans. $(x^2 - 1)(x + 5)$.

7. Of $10x(x + y)$, $8y(x - y)$, and $5(x^2 - y^2)$.
Ans. $40xy(x^2 - y^2)$.

8. Of $18x^4(x - y)$, $25x^3(x - y)^2$, and $12x^5(x - y)^3$.
Ans. $900x^5(x - y)^3$.

9. Of $x^3 - 1$, and $x^2 + x - 2$.
Ans. $x^4 + 2x^3 - x - 2$.

10. Of $6x^2 - x - 1$, and $2x^2 + 3x - 2$.
Ans. $6x^3 + 11x^2 - 3x - 2$.

11. Of $a - x$, $a^2 - x^2$, and $a^3 - x^3$.
Ans. $a^4 + a^3x - ax^3 - x^4$.

12. Of $3x^2 - 11x + 6$, $2x^2 - 7x + 3$, $6x^2 - 7x + 2$.
Ans. $6x^3 - 25x^2 + 23x - 6$.

CHAPTER IV.

FRACTIONS.

I. DEFINITIONS AND PRINCIPLES.

Definitions.

53. If the unit 1 is divided into any number of equal parts, each part is called a **fractional unit.** Thus, $\frac{1}{2}, \frac{1}{4}, \frac{1}{7}, \frac{1}{b}$, are fractional units.

54. A **fraction** is a fractional unit, or a collection of fractional units. Thus, $\frac{1}{2}, \frac{3}{4}, \frac{5}{7}, \frac{a}{b}$, are fractions.

55. Every fraction is composed of two parts: the *denominator*, which shows into how many parts the unit 1 is divided; and the *numerator*, which shows how many of these parts are taken. Thus, in the fraction $\frac{a}{b}$, the denominator shows that 1 is divided into b equal parts, and the numerator shows that a of these parts are taken. The fractional unit, in all cases, is equal to the reciprocal of the denominator.

56. A **decimal fraction** is one whose denominator is some power of 10. In such fractions, the denominator is usually indicated by a decimal point, which signifies that the denominator is equal to 1, followed by as many 0's as there are decimal places. Thus, the expressions, .034, .0079, are decimal fractions, equivalent to the fractions, $\frac{34}{1000}$, $\frac{79}{10000}$.

57. An **entire quantity**, is one which contains no fractional part. Thus, 7, 11, a^3x, $4x^2 - 3y$, are *entire quantities.*

An entire quantity may be regarded as a fraction whose denominator is 1. Thus, $7 = \frac{7}{1}$, $ab = \frac{ab}{1}$.

58. A **mixed quantity**, is a quantity containing both entire and fractional parts. Thus, $7\frac{4}{10}$, $8\frac{3}{7}$, $a + \frac{bx}{c}$, are mixed quantities.

Principles.

59. Let $\frac{a}{b}$ denote any fraction, and let q be any quantity whatever. From preceding definitions, $\frac{a}{b}$ indicates that the fractional unit $\frac{1}{b}$ is taken a times; also, $\frac{aq}{b}$ indicates that the same fractional unit is taken aq times, that is,

$$\frac{aq}{b} = \frac{a}{b} \times q, \quad \text{or} \quad q \times \frac{a}{b} = \frac{aq}{b}.$$

Hence, the following principle:

1°. *Multiplying the numerator of a fraction by any quantity is equivalent to multiplying the fraction by that quantity.*

We infer from what precedes that we may multiply any quantity by a fraction, by first multiplying that quantity by the numerator of the fraction, and then dividing the result by the denominator.

It is a principle of division that the same result will be obtained if we divide the quantity a by the product of two factors pq, that would be obtained by dividing it by one of the factors, p, and then dividing that result by the other factor, q, that is,

$$\frac{a}{pq} = \left(\frac{a}{p}\right) \div q, \quad \text{or} \quad \frac{a}{pq} = \left(\frac{a}{q}\right) \div p.$$

Hence, the following principle:

2°. *Multiplying the denominator of a fraction by any quantity, is equivalent to dividing the fraction by that quantity.*

Since the operations of multiplication and division are the reverse of each other, we have, from what has been shown, the following principles:

3°. *Dividing the numerator of a fraction by any quantity, is equivalent to dividing the fraction by that quantity.*

4°. *Dividing the denominator of a fraction by any quantity, is equivalent to multiplying the fraction by that quantity.*

Because a quantity may be multiplied by any quantity, and the result divided by the same quantity without changing its value, we have the following principle:

5°. *Both terms of a fraction may be multiplied, or divided, by any quantity without changing the value of the fraction.*

II. TRANSFORMATIONS OF FRACTIONS.

Definitions.

60. The **transformation** of a quantity is the operation of changing its form without altering its value.

FIRST TRANSFORMATION. *To reduce an entire quantity to a fractional form having a given denominator.*

61. Let a be the entire quantity, and b the given denominator. We have, evidently,

$$a = \frac{ab}{b}.$$

Hence, the following

RULE.

Multiply the entire quantity by the given denominator and write the product over that denominator.

EXAMPLES.

1. Reduce $3x^2$ to a fractional form whose denominator is b^2. *Ans.* $\frac{3b^2x^2}{b^2}$.

2. Reduce $13x - 2y$ to a fractional form whose denominator is $x - y$. *Ans.* $\frac{13x^2 - 15xy + 2y^2}{x - y}$.

SECOND TRANSFORMATION. *To reduce a fraction to its lowest terms.*

62. A fraction is in its *lowest terms*, when its numerator and denominator contain no common factors.

From principle 5°, Art. 59, we deduce the following

RULE.

Resolve both terms of the fraction into their prime factors; then strike out all that are common to both.

The same result is attained by dividing both terms of the fraction by their greatest common divisor.

EXAMPLES.

1. Reduce $\frac{14ax}{21ay}$ to its lowest terms.

Factoring both terms, we have,

$$\frac{14ax}{21ay} = \frac{2x \times 7a}{3y \times 7a};$$

Striking out the common factor $7a$, we have,

$$\frac{14ax}{21ay} = \frac{2x}{3y}. \quad \textit{Ans.}$$

2. Reduce the fraction $\frac{3a^2 - 3b^2}{4a - 4b}$, to its lowest terms.

We have,

$$\frac{3a^2 - 3b^2}{4a - 4b} = \frac{3(a+b)(a-b)}{4(a-b)} = \frac{3(a+b)}{4}. \quad \textit{Ans.}$$

Reduce the following fractions to their lowest terms:

3. $\frac{16abx^2}{24a^2b^2x}$. *Ans.* $\frac{2x}{3ab}$.

4. $\frac{12a^2cd}{16abc}$. *Ans.* $\frac{3ad}{4b}$

5. $\frac{45x^3y^3z}{36abx^2y^2z}$. *Ans.* $\frac{5xy}{4ab}$.

6. $\dfrac{2(x^2 - y^2)}{x^2 - 2xy + y^2}$. *Ans.* $\dfrac{2(x + y)}{x - y}$.

7. $\dfrac{5(a^2 - x^2)}{10(a - x)}$. *Ans.* $\frac{1}{2}(a + x)$.

8. $\dfrac{x^2 - a^2}{x^2 + 2ax + a^2}$. *Ans.* $\dfrac{x - a}{x + a}$.

9. $\dfrac{3ax^2 - 3a^2x}{2x^2y - 2axy}$. *Ans.* $\dfrac{3a}{2y}$.

10. $\dfrac{3x^2 - 6x}{2xy - 4y}$. *Ans.* $\dfrac{3x}{2y}$.

11. $\dfrac{x^2 + 2ax + a^2}{3(x^2 - a^2)}$. *Ans.* $\dfrac{x + a}{3(x - a)}$.

12. $\dfrac{x^2 + x - 2}{2x^3 - 3x + 1}$. *Ans.* $\dfrac{x + 2}{2x^2 + 2x - 1}$.

The last example is solved by finding the greatest common divisor of the two terms, which is $x - 1$, and then dividing both by it. The following examples may be solved in the same manner:

13. $\dfrac{x^2 - 9}{x^2 - x - 12}$. *Ans.* $\dfrac{x - 3}{x - 4}$.

14. $\dfrac{x^2 - 2ax + a^2}{7x^2y - 10axy + 3a^2y}$. *Ans.* $\dfrac{x - a}{(7x - 3a)y}$.

15. $\dfrac{12x^2 - 15xy + 3y^2}{6x^3 - 6x^2y + 2xy^2 - 2y^3}$. *Ans.* $\dfrac{12x - 3y}{6x^2 + 2y^2}$.

THIRD TRANSFORMATION. *To reduce a fraction to the form of a mixed quantity.*

63. From the nature of the operation of division, we have the following

RULE.

Perform the indicated division, continuing the operation as far as possible; then write the remainder over the denominator, and add the result to the quotient found.

EXAMPLES.

1. $\frac{a^2 + x^2}{a + x} = a - x + \frac{2x^2}{a + x}.$

2. $\frac{x^2 + x - 4}{x + 2} = x - 1 - \frac{2}{x + 2}.$

3. $\frac{a^3 + x^3}{a^2 + 2ax + x^2} = a - 2x + \frac{3x^3}{a + x}.$

4. $\frac{x^2 + a^2 + 3 - 2ax}{x - a} = x - a + \frac{3}{x - a}.$

5. $\frac{30 - 11x - 44x^2 + 32x^3}{15 + 17x - 4x^2} = 2 - 3x + \frac{5x^2}{5 - x}.$

6. $\frac{x^2 + 3x - 25}{x - 4} = x + 7 + \frac{3}{x - 4}.$

7. $\frac{2y^4 + 19y^2 + 35}{y^3 - 3y^2 + 7y - 21} = 2y + 6 + \frac{23}{y - 3}.$

8. $\frac{4x - x^2 + 3 - y}{4 - x} = x + \frac{3 - y}{4 - x}.$

Fourth Transformation. *To reduce a mixed quantity to a fractional form.*

64. By reversing the rule of Art. 63, we have the following

RULE.

Multiply the entire part by the denominator of the fraction; to the product annex the numerator with the sign of the fraction, and under the result write the denominator.

EXAMPLES.

1. $a + \frac{b}{c} = \frac{ac + b}{c}$.

2. $a + \frac{ax}{a - x} = \frac{a(a - x) + ax}{a - x} = \frac{a^2}{a - x}$.

3. $1 + \frac{c}{x - y} = \frac{x - y + c}{x - y}$.

4. $1 + \frac{a^2 + b^2 - c^2}{2ab} = \frac{(a + b)^2 - c^2}{2ab}$.

5. $a + \frac{ac + d}{c + d} = \frac{2ac + ad + d}{c + d}$.

6. $x + y + \frac{x^2 + y^2}{x - y} = \frac{2x^2}{x - y}$.

7. $ab + cd + \frac{abc - c^2d - 2cd^2}{c + 2d} = \frac{2ab(c + d)}{c + 2d}$.

8. $xy - ab - \frac{2xy^2 - 2aby}{x + y} = \frac{(xy - ab)(x - y)}{x + y}$.

Fifth Transformation. *To reduce fractions to a common denominator.*

65. Take the fractions $\frac{a}{b}$, $\frac{m}{n}$, and $\frac{x}{y}$. Since both terms of a fraction may be multiplied by the same quantity without altering its value (principle 5°), we multiply both terms of the first fraction by ny, both terms of the second fraction by by, and both terms of the third fraction by bn, which gives

$$\frac{any}{bny}, \quad \frac{bmy}{bny}, \quad \text{and} \quad \frac{bnx}{bny};$$

these are equivalent to the given fractions and have a common denominator. In the same way any other group of fractions may be reduced to equivalent fractions having a common denominator; hence, the following

RULE.

Find the product of all the denominators for a common denominator; multiply the numerator of each fraction by the product of the denominators of all the others for new numerators.

EXAMPLES.

1. Reduce $\frac{2x}{3}$, $\frac{3x^2}{y}$, and $\frac{y}{x}$, to a common denominator.

Ans. $\frac{2x^2y}{3xy}$, $\frac{9x^3}{3xy}$, and $\frac{3y^2}{3xy}$.

2. Reduce $\frac{3x}{4a^2}$, $\frac{2y}{3a}$, and $\frac{z}{6a^3}$, to a common denominator.

Ans. $\frac{54a^4x}{72a^6}$, $\frac{48a^5y}{72a^6}$, and $\frac{12a^3z}{72a^6}$.

Fractions may always be reduced to a common denominator, by the preceding rule; but, if the denominators have any common factors, the denominator thus found will not be the least common denominator.

Taking the example just given, we see that the least common multiple of the given denominators is $12a^3$. Dividing this by each denominator separately, and then multiplying both terms of each fraction by the corresponding quotient, we have,

$$\frac{9ax}{12a^3},\ \frac{8a^2y}{12a^3},\ \text{and}\ \frac{2z}{12a^3},$$

which are equivalent to the given fractions, and have the least common denominator. In like manner we may reduce any group of fractions to their least common denominator; hence, the following

RULE.

Find the least common multiple of all the denominators, for a common denominator; divide this by each denominator, separately, and multiply the quotients by the corresponding numerators for new numerators.

If there are any entire quantities, they may be regarded as fractions whose denominators are equal to 1.

Reduce each of the following groups of fractions to their least common denominators:

3. $\frac{x+a}{b},\ \frac{a}{b},\ \text{and}\ \frac{a-x}{a}$.

Ans. $\frac{a(x+a)}{ab},\ \frac{a^2}{ab},\ \text{and}\ \frac{b(a-x)}{ab}$.

4. $\frac{m}{4a(a+x)},\ \text{and}\ \frac{n}{4(a^2-x^2)}$.

Ans. $\frac{m(a-x)}{4a(a^2-x^2)},\ \text{and}\ \frac{na}{4a(a^2-x^2)}$.

5. $\frac{3x}{4}$, $\frac{4}{6}$, and $\frac{12x^2}{15}$. *Ans.* $\frac{45x}{60}$, $\frac{40}{60}$, and $\frac{48x^2}{60}$.

6. $\frac{2b}{15}$, $\frac{3c}{5}$, and $\frac{4d}{25}$. *Ans.* $\frac{10b}{75}$, $\frac{45c}{75}$, and $\frac{12d}{75}$.

7. a, $\frac{3b^2}{4}$, and $\frac{5c^3}{6}$. *Ans.* $\frac{12a}{12}$, $\frac{9b^2}{12}$, and $\frac{10c^3}{12}$.

8. $\frac{x}{1-x}$, $\frac{x^2}{(1-x)^2}$, and $\frac{x^3}{(1-x)^3}$.

Ans. $\frac{x(1-x)^2}{(1-x)^3}$, $\frac{x^2(1-x)}{(1-x)^3}$, and $\frac{x^3}{(1-x)^3}$.

9. $3bx$, $\frac{a}{a+x}$, $\frac{b}{a^2-x^2}$, and $\frac{c}{x}$.

Ans. $\frac{3bx^2(a^2-x^2)}{x(a^2-x^2)}$, $\frac{ax(a-x)}{x(a^2-x^2)}$, $\frac{bx}{x(a^2-x^2)}$, $\frac{c(a^2-x^2)}{x(a^2-x^2)}$.

10. $\frac{cx}{a-x}$, $\frac{dx^2}{a+x}$, and $\frac{x^3}{a+x}$.

Ans. $\frac{cx(a+x)}{a^2-x^2}$, $\frac{dx^2(a-x)}{a^2-x^2}$, and $\frac{x^3(a-x)}{a^2-x^2}$.

11. $\frac{4}{c-x}$, $\frac{5}{x^2}$, and $\frac{6}{x^3}$.

Ans. $\frac{4x^3}{x^3(c-x)}$, $\frac{5x(c-x)}{x^3(c-x)}$, and $\frac{6(c-x)}{x^3(c-x)}$.

12. 4, $\frac{5}{a^2-x^2}$, $\frac{6}{a^2+x^2}$, and $\frac{7}{y}$.

Ans. $\frac{4y(a^4-x^4)}{y(a^4-x^4)}$, $\frac{5y(a^2+x^2)}{y(a^4-x^4)}$, $\frac{6y(a^2-x^2)}{y(a^4-x^4)}$, and $\frac{7(a^4-x^4)}{y(a^4-x^4)}$.

In what follows, we shall suppose entire and mixed quantities to be reduced to fractional forms, and all will be treated together as fractions.

III. ADDITION OF FRACTIONS.

Demonstration of the Rule.

66. Fractions can be added when they have a common unit, that is, when they have a common denominator. In that case, the numerator of each fraction indicates the number of times the common fractional unit is taken, in that fraction; and the sum of the numerators indicates how many times that unit is taken in the entire collection; hence, the following rule for addition of fractions:

RULE.

Reduce the fractions to a common denominator; then add the numerators for a new numerator, and write their sum over the common denominator.

EXAMPLES.

1. Find the sum of a, $\frac{1}{a}$, $\frac{1}{2b}$, and $\frac{3x}{4a^2}$.

Reducing to a common denominator, we have,

$$\frac{4a^3b}{4a^2b},\quad \frac{4ab}{4a^2b},\quad \frac{2a^2}{4a^2b},\quad \frac{3bx}{4a^2b};$$

hence, the sum is $\frac{(4a^3 + 4a + 3x)b + 2a^2}{4a^2b}$.

Find the sums of the following quantities.

2. $\frac{2}{a^2b^3}$, $\frac{3}{a^3b^2}$, and $\frac{4}{a^3b^3}$ *Ans.* $\frac{2a + 3b + 4}{a^3b^3}$.

3. $\frac{2a}{3x^2}$, $\frac{a + 2x}{4x}$, and $\frac{a}{6x}$. *Ans.* $\frac{6x^2 + 5ax + 8a}{12x^2}$.

4. $\frac{x+y}{2}$, and $\frac{x-y}{2}$. *Ans.* x.

5. $\frac{2x}{3}$, $\frac{3x}{5}$, and $\frac{5x}{7}$. *Ans.* $x + \frac{103}{105}x$.

6. $\frac{2x}{1-x^2}$, and $\frac{1}{1+x}$. *Ans.* $\frac{1}{1-x}$.

7. $\frac{2}{(x-1)^3}$, $\frac{3}{(x-1)^2}$, and $\frac{4}{x-1}$.
Ans. $\frac{4x^2-5x+3}{(x-1)^3}$.

8. $\frac{a}{(1+a)(a+x)}$, and $\frac{x}{(1-x)(a+x)}$.
Ans. $\frac{1}{(1+a)(1-x)}$.

9. $\frac{1}{4(1+a)}$, $\frac{1}{4(1-a)}$, and $\frac{1}{2(1-a^2)}$.
Ans. $\frac{1}{1-a^2}$.

10. $\frac{3x-4y}{7}$, $\frac{-2x+y+1}{3}$, and $\frac{15x-4}{12}$.
Ans. $\frac{85x-20y}{84}$.

IV. SUBTRACTION OF FRACTIONS.

Demonstration of the Rule.

67. If the minuend and subtrahend have the same unit, the numerator of the minuend diminished by that of the subtrahend will indicate the number of times this unit is contained in the difference; hence the following

RULE.

Reduce the fractions to a common denominator; then subtract the numerator of the subtrahend from that of the minuend for a new numerator, and write the remainder over the common denominator.

EXAMPLES.

1. From $\frac{a+2x}{a-2x}$, subtract $\frac{a-2x}{a+2x}$.

By the rule,

$$\frac{a+2x}{a-2x}-\frac{a-2x}{a+2x}=\frac{(a+2x)^2}{a^2-4x^2}-\frac{(a-2x)^2}{a^2-4x^2}=\frac{8ax}{a^2-4x^2}.$$

2. From $4a+\frac{2a}{c}$, subtract $2a-\frac{a-3b}{c}$.

Ans. $2a+\frac{3(a-b)}{c}$.

3. From $\frac{5x+3y}{4}$, subtract $\frac{x-2y}{5}$.

Ans. $\frac{21x+23y}{20}$.

4. From $\frac{a}{a-x}$, subtract $\frac{x}{a+x}$. *Ans.* $\frac{a^2+x^2}{a^2-x^2}$.

5. From $\frac{x+y}{x-y}$, subtract $\frac{x-y}{x+y}$. *Ans.* $\frac{4xy}{x^2-y^2}$.

6. From $a+\frac{a-x}{a(a+x)}$, subtract $\frac{a+x}{a(a-x)}$.

Ans. $a-\frac{4x}{a^2-x^2}$.

7. From $3x+\frac{11x-10}{15}$, subtract $2x+\frac{3x-5}{7}$.

Ans. $\frac{137x+5}{105}$.

8. From $\frac{1}{y-z}$, subtract $\frac{1}{y^2-z^2}$. *Ans.* $\frac{y+z-1}{y^2-z^2}$.

V. MULTIPLICATION OF FRACTIONS.

Demonstration of the Rule.

68. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions. It was shown in Art. 59, that any quantity may be multiplied by a fraction, by first multiplying by the numerator, and then dividing the result by the denominator. To multiply $\frac{a}{b}$ by $\frac{c}{d}$, we first multiply by c, giving $\frac{ac}{b}$, (principle 1°); then, we divide this result by d, which is done by multiplying the denominator by d, (principle 2°); this gives for the product, $\frac{ac}{bd}$; that is,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Hence, we have the following

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

EXAMPLES.

1. Multiply $\frac{7x}{5y}$, by $\frac{3a}{4c}$.

The product of the numerators is $21ax$, and of the denominators $20cy$; hence, by the rule,

$$\frac{7x}{5y} \times \frac{3a}{4c} = \frac{21ax}{20cy}.$$

2. Multiply $\frac{2x}{x-y}$, by $\frac{x^2-y^2}{3}$.

We have, by the rule,

$$\frac{2x}{x-y} \times \frac{x^2-y^2}{3} = \frac{2x(x+y)(x-y)}{3(x-y)} = \frac{2x(x+y)}{3}.$$

In the above example, after indicating the multiplication, we factored both numerator and denominator, and then struck out the common factor before performing the operation. It will, in general, be found expedient to follow this method.

3. Multiply $2\left(\frac{x+y}{x-y}\right)$, by $\frac{x^2-y^2}{x^2+2xy+y^2}$.

Indicating the operation, factoring, &c., we have,

$$\frac{2(x+y)}{x-y} \times \frac{x^2-y^2}{x^2+2xy+y^2} = \frac{2(x+y)(x-y)(x+y)}{(x-y)(x+y)(x+y)} = 2.$$

Find the products of the following groups of quantities:

4. $\frac{3x^2y}{4a}$, and $\frac{2a^2b}{c}$. *Ans.* $\frac{3abx^2y}{2c}$.

5. $\frac{7abf}{3cd}$, and $\frac{4x^2y^2}{3ab^2}$. *Ans.* $\frac{28fx^2y^2}{9bcd}$.

6. $\frac{7x+6}{3}$, and $\frac{2x}{5}$. *Ans.* $\frac{14x^2+12x}{15}$.

7. $\frac{2}{x-y}$, and $\frac{x^2-y^2}{a}$. *Ans.* $\frac{2(x+y)}{a}$.

8. $\frac{ab}{4-x}$, and $\frac{3x^2}{a^2}$. *Ans.* $\frac{3bx^2}{a(4-x)}$.

9. $\frac{x^2-4}{3}$, and $\frac{4x}{x+2}$ *Ans.* $\frac{4x(x-2)}{3}$.

10. $a+\frac{b}{x}$, and $b+\frac{a}{x}$. *Ans.* $\frac{(ax+b)(bx+a)}{x^2}$.

11. $3+\frac{x}{4}$, and $x+\frac{4}{x}$. *Ans.* $\frac{(12+x)(x^2+4)}{4x}$

12. $\frac{(a+b)^2}{2x}$, and $\frac{4x^2}{a+b}$. *Ans.* $2x(a+b)$

13. $\frac{3x^2+1}{2}$, and $\frac{2y}{3}$. *Ans.* $\frac{y(3x^2+1)}{3}$.

14. $\frac{(x-1)^2}{y^3}$, and $\frac{(x+1)y^2}{x-1}$. *Ans.* $\frac{x^2-1}{y}$.

15. $m+\frac{1}{m}-1$, and $m+\frac{1}{m}+1$.

Ans. $m^2+1+\frac{1}{m^2}$.

16. $x-\frac{y^2}{x}$, and $\frac{x}{y}+\frac{y}{x}$. *Ans.* $\frac{x^4-y^4}{x^2y}$.

17. $\frac{x(a-x)}{a^2+2ax+x^2}$, and $\frac{a(a+x)}{a^2-2ax+x^2}$.

Ans. $\frac{ax}{a^2-x^2}$.

18. $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{x(a+x)}$, and $a+\frac{ax}{a-x}$.

Ans. $\frac{a^2(a-b)}{x}$.

19. $x+\frac{2xy}{x-y}$, and $x-\frac{2xy}{x+y}$. *Ans.* x^2.

20. $\frac{a^3-x^3}{a^3+x^3}$, $\frac{a^2-x^2}{a^2+x^2}$, $\frac{a-x}{a+x}$, and $\frac{a^2-ax+x^2}{a^2+ax+x^2}$.

Ans. $\frac{(a-x)^3}{a^3+a^2x+ax^2+x^3}$.

21. x^2+x+1, and $\frac{1}{x^2}-\frac{1}{x}+1$. *Ans.* $x^2+1+\frac{1}{x^2}$.

22. $x + 1 + \frac{1}{x}$, and $x - 1 + \frac{1}{x}$. *Ans.* $x^2 + 1 + \frac{1}{x^2}$.

23. $\frac{2a - b}{4a}$, and $\frac{6a - 2b}{b^2 - 2ab}$. *Ans.* $\frac{b - 3a}{2ab}$.

24. $\frac{a}{a + b} + \frac{b}{a - b}$, and $\frac{a}{a + c} - \frac{b}{b + c}$.

Ans. $\frac{(a^2 + b^2)c}{(a + b)(a + c)(b + c)}$.

VI. DIVISION OF FRACTIONS.

Demonstration of the Rule.

69. Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$. From the nature of division, it follows that dividing one quantity by another is equivalent to multiplying the former by the reciprocal of the latter; but the reciprocal of the fraction $\frac{c}{d}$ is $\frac{d}{c}$, (Art. 14), that is, the reciprocal of the fraction is found by inverting its terms; performing the operation of inverting and multiplying, we have:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc};$$

hence, we have the following rule for dividing one fraction by another:

RULE.

Invert the terms of the divisor, and multiply the dividend by the resulting fraction.

After indicating the operation of multiplication strike out all the factors that are common to the numerator and denominator.

EXAMPLES.

1. Divide $\dfrac{3a^2}{a^2-b^2}$, by $\dfrac{a}{a+b}$.

By the rule, we have,

$$\frac{3a^2}{a^2-b^2} \div \frac{a}{a+b} = \frac{3a^2 \times (a+b)}{(a^2-b^2)a} = \frac{3a^2(a+b)}{a(a+b)(a-b)} = \frac{3a}{a-b}.$$

Perform the following divisions:

2. $\dfrac{3x}{2x-2}$, by $\dfrac{2x}{x-1}$. *Ans.* $\frac{3}{4}$.

3. $\dfrac{(x+y)^2}{x-y}$, by $\dfrac{x+y}{(x-y)^2}$. *Ans.* x^2-y^2.

4. $x+\dfrac{x}{x-1}$, by $x-\dfrac{x}{x-1}$. *Ans.* $\dfrac{x}{x-2}$.

5. $\dfrac{x^3-3ax^2+3a^2x-a^3}{x+a}$, by $\dfrac{x-a}{x+a}$. *Ans.* $(x-a)^2$.

6. $\dfrac{x^4-y^4}{a^3+b^3}$, by $\dfrac{x-y}{a^2-ab+b^2}$. *Ans.* $\dfrac{x^3+x^2y+xy^2+y^3}{a+b}$.

7. $x^2+2+\dfrac{1}{x^2}$, by $\dfrac{x}{a}+\dfrac{1}{ax}$. *Ans.* $\dfrac{ax^2+a}{x}$.

8. $\dfrac{2ax+x^2}{c^3-x^3}$, by $\dfrac{x}{c-x}$. *Ans.* $\dfrac{2a+x}{c^2+cx+x^2}$.

9. $x+y+\dfrac{x^2}{y}$, by $x+y+\dfrac{y^2}{x}$. *Ans.* $\dfrac{x}{y}$.

10. $\dfrac{x^2-9}{x^2+4x+4}$, by $\dfrac{x+3}{x+2}$. *Ans.* $\dfrac{x-3}{x+2}$.

11. 1, by $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. *Ans.* $\frac{abc}{ab+ac+bc}$.

12. $a^4-\frac{1}{a^4}$, by $a-\frac{1}{a}$. *Ans.* $a^3+a+\frac{1}{a}+\frac{1}{a^3}$.

MISCELLANEOUS EXAMPLES.

Simplify the following expressions:

1. $\frac{a^2+ab+b^2}{2}+\frac{a^2-ab+b^2}{2}$. *Ans.* a^2+b^2.

2. $\frac{1}{a-1}-\frac{2a}{a^2+1}+\frac{1}{a+1}$. *Ans.* $\frac{4a}{a^4-1}$.

3. $\frac{y-1}{2}+\frac{y-2}{3}+\frac{y+7}{6}$. *Ans.* y.

4. $\frac{2x}{x^2-y^2}+\frac{1}{x+y}-\frac{1}{x-y}$. *Ans.* $\frac{2}{x+y}$.

5. $\frac{x-y}{y}+\frac{2x}{x-y}-\frac{x^3+x^2y}{x^2y-y^3}$. *Ans.* $\frac{y}{x-y}$.

6. $\left\{x+\frac{y-x}{1+xy}\right\}\div\left\{1-x\frac{y-x}{1+xy}\right\}$. *Ans.* y.

7. $\frac{3}{2y-3}-\frac{2y+15}{4y^2+9}-\frac{2}{2y+3}$. *Ans.* $\frac{18(2y+15)}{16y^4-81.}$.

8. $\frac{x}{1-x}-\frac{x^2}{(1-x)^2}+\frac{x^3}{(1-x)^3}$. *Ans.* $x+\frac{x^4}{(1-x)^3}$.

9. $\left\{\frac{x+2y}{x+y}+\frac{x}{y}\right\}\div\left\{\frac{x+2y}{y}-\frac{x}{x+y}\right\}$.

Ans. 1.

10. $\dfrac{x^2 - 9x + 20}{x^2 - 6x} \times \dfrac{x^2 - 13x + 42}{x^2 - 5x}$.

Ans. $\dfrac{(x - 4)\,(x - 7)}{x^2}$.

11. $\dfrac{x^{3n}}{x^n - 1} - \dfrac{x^{2n}}{x^n + 1} + \dfrac{1}{x^n + 1} - \dfrac{1}{x^n - 1}$.

Ans. $x^{2n} + 2$.

12. $\dfrac{a^2 - ax}{bc + bx} \times \dfrac{4(a + x)}{3(c - x)}$. *Ans.* $\dfrac{4a(a^2 - x^2)}{3b(c^2 - x^2)}$.

13. $\dfrac{a}{b} - \dfrac{a^2 - b^2}{b^2}x + \dfrac{a(a^2 - b^2)x^2}{b^2(b + ax)}$. *Ans.* $\dfrac{a + bx}{b + ax}$.

14. $\dfrac{a^4 - 2a^2x^2 + x^4}{a^3x + ax^3} \div \left\{\dfrac{a + x}{a} \times \dfrac{a - x}{x}\right\}$.

Ans. $\dfrac{a^2 - x^2}{a^2 + x^2}$.

15. $\left\{a + \dfrac{2ax - 1}{b}\right\} \div \dfrac{x - a}{ax + 1}$.

Ans. $\dfrac{a^2x(b + 2x) + a(x + b) - 1}{b(x - a)}$.

CHAPTER V.

I. EQUATIONS OF THE FIRST DEGREE.

Definitions.

70. AN **equation** is an expression of equality between two quantities. Thus,

$$x = b + c,$$

is an equation, expressing the fact that x is equal to the sum of b and c.

71. Every equation is composed of two parts, connected by the sign of equality. These parts are called **members**: the part on the left of the sign of equality, is called the **first member**; that on the right, the **second member.** Thus, in the equation,

$$x + y = a - c,$$

$x + y$, is the first member, and $a - c$, the second member.

Either member of an equation may be 0; in this case the algebraic sum of the quantities in the other member is 0.

Classification.

72. Equations are divided into two classes: those containing *but one* unknown quantity, and those containing *more than one* unknown quantity. Each of these classes is subdivided into **degrees**. In the first class, the degree is determined by the exponent of the highest power of the unknown quantity, in any term; in the second class, the degree is determined by the highest sum of the exponents of the unknown quantities, in any term.

Thus,

$$bx = c, \qquad bx + cy = d,$$

are equations of the *first degree;*

$$x^2 + 2px = q, \qquad x^2 + axy + y^2 = m,$$

are equations of the *second degree;*

$$ax^3 + bx^2 + cx = d, \qquad x^3 + 2x^2y + 3yx + 4y = 5,$$

are equations of the *third degree;*

$$x^n + px^{n-1} + qx^{n-2} = s, \qquad x^{n-2}y^2 + ax^{n-3}y + bxy^{n-1} = d,$$

are equations of the *n^{th} degree.*

We shall first consider equations of the first degree, containing but one unknown quantity.

Definitions.

73. The **transformation** of an equation, is the operation of changing its form, without destroying the equality of its members.

74. The **solution** of an equation, is the operation of finding such a value for the unknown quantity, as will *satisfy* the equation; that is, such a value as, being substituted for the unknown quantity, will render the two members equal. This value is called a **root** of the equation.

75. An **axiom** is a self-evident proposition.

Axioms.

76. The solution of an equation is effected by successive transformations, which transformations depend on the following axioms:

1°. If equal quantities are added to both members of an equation, the equality will not be destroyed.

2°. If equal quantities are subtracted from both members of an equation, the equality will not be destroyed.

3°. If both members of an equation are multiplied by the same quantity, the equality will not be destroyed.

4°. If both members of an equation are divided by the same quantity, the equality will not be destroyed.

5°. Like powers of the two members of an equation are equal.

6°. Like roots of the two members of an equation are equal.

Principal Transformations.

77. Two principal transformations are employed in the solution of equations of the first degree: **clearing of fractions**, and **transposing.**

FIRST TRANSFORMATION. *Clearing of fractions.*

78. Take the equation,

$$\frac{4x}{5} + \frac{6}{10} = \frac{3}{4}.$$

The least common multiple of all the denominators is 20. If we multiply both members of the equation by 20, (axiom 3°), each term can be reduced to an entire form, giving,

$$16x + 12 = 15.$$

In the same manner, any equation may be transformed; hence, for clearing of fractions, we have the following

RULE.

Find the least common multiple of the denominators, and multiply both members of the equation by it, reducing fractional to entire terms.

The reduction will be effected, if we divide the least common multiple by each denominator, and then multiply the quotient by the corresponding numerator, dropping the denominator

EXAMPLES.

1. Clear the equation, $\frac{x}{4} - \frac{2x}{3} = \frac{5}{6}$, of fractions.

The least common multiple of the denominators is 12. Multiplying both members by 12 (axiom 3°), and reducing to entire terms, we have,

$$3x - 8x = 10.$$

2. Clear the equation, $\frac{7x}{5} - \frac{2x}{3} = \frac{5}{2}$, of fractions.

Ans. $42x - 20x = 75$.

Clear the following equations of fractions:

3. $\frac{x}{7} - \frac{3x}{2} + \frac{x}{4} = 5$. *Ans.* $4x - 42x + 7x = 140$.

4. $\frac{13x}{12} + \frac{4}{3} = 6x.$ *Ans.* $13x + 16 = 72x.$

5. $\frac{x-4}{3} + \frac{x-2}{6} = \frac{5}{3}.$ *Ans.* $2x - 8 + x - 2 = 10.$

6. $-\frac{x-4}{3} - \frac{x-2}{6} = \frac{5}{3}.$ *Ans.* $-2x + 8 - x + 2 = 10.$

7. $\frac{x-3}{12} - \frac{3x-4}{21} = 8.$

Ans. $7x - 21 - 12x + 16 = 672.$

8. $\frac{x}{3-x} + 4 = \frac{3}{5}.$ *Ans.* $5x + 60 - 20x = 9 - 3x.$

SECOND TRANSFORMATION. *Transposing.*

79. **Transposition** is the operation of changing a term from one member to the other, without destroying the equality.

Take the equation,

$$3x + 4 = 5 - 6x.$$

If we add the quantity $6x - 4$, to both members of the equation, (axiom 1°), we shall have,

$$3x + 4 + 6x - 4 = 5 - 6x + 6x - 4;$$

which reduces to

$$3x + 6x = 5 - 4.$$

Comparing this with the given equation, we see that 4 has been transposed to the second member, and $-6x$ to the first member, by changing their signs. In like manner, any term may be transposed; hence, the following rule for transposing:

RULE.

Any term may be transposed from one member to the other, if we change its sign.

Transpose the unknown terms to the first, and the known terms to the second members, in the following

EXAMPLES.

1. $13x + 16 = 7x + 20$. *Ans.* $13x - 7x = 20 - 16$.
2. $2x - 8 = 10 - x$. *Ans.* $2x + x = 10 + 8$.
3. $7x - 21 = -12x + 1$. *Ans.* $7x + 12x = 1 + 21$.
4. $5x + 60 = 9 - 3x$. *Ans.* $5x + 3x = 9 - 60$.
5. $3x - 7 = -x - 8$. *Ans.* $3x + x = 7 - 8$.

Method of Solving Equations.

80. Take the equation,

$$\frac{2x}{3} + 7 = \frac{5(x-1)}{10} + 8.$$

Clearing of fractions, (Art. 78), and performing the operations indicated, we have,

$$20x + 210 = 15x - 15 + 240.$$

Transposing all the unknown terms to the first member, and the known terms to the second member, (Art. 79), we have,

$$20x - 15x = 240 - 15 - 210.$$

Reducing the terms in the two members, we have

$$5x = 15.$$

Dividing both members by the coefficient of x, we have,

$$x = 3.$$

In the same way, all equations of the first degree, containing but one unknown quantity, may be solved; hence, the following

RULE.

I. Clear the equation of fractions, and perform all the indicated operations.

II. Transpose all the unknown terms to the first member, and all the known terms to the second member.

III. Reduce all the terms in the first member to a single term, one factor of which is the unknown quantity; the other factor will be the algebraic sum of its coefficients.

IV. Divide both members by the coefficient of the unknown quantity: the second member will be the value of the unknown quantity.

EXAMPLES.

1. Solve the equation,

$$5x - \frac{6x + 3}{11} = \frac{7x + 15}{2} - 3.$$

Clearing of fractions, $110x - 12x - 6 = 77x + 165 - 66$;
transposing and reducing, $21x = 105$;
dividing by 21, $x = 5$.

2. Solve the equation,

$$\frac{b}{ax} - a^2 = b^2 - \frac{a}{bx}.$$

Clearing of fractions, $b^2 - a^3bx = ab^3x - a^2$;
transposing and reducing, $-(a^3b + ab^3)x = -(a^2 + b^2)$;
dividing by $-(a^3b + ab^3)$, $x = \dfrac{a^2 + b^2}{a^3b + ab^3} = \dfrac{1}{ab}$.

Solve the following equations:

3. $\dfrac{3x - 4}{2} = \dfrac{x}{2} + \dfrac{x}{4} - \dfrac{1}{2}$. *Ans.* $x = 2$.

4. $\dfrac{x}{8} - 1 + \dfrac{x}{12} - \dfrac{x + 5}{4} = -\dfrac{11}{4}$. *Ans.* $x = 12$.

5. $\dfrac{x + a}{b} - \dfrac{x}{a} = 1$. *Ans.* $x = -a$.

6. $\dfrac{x - 1}{2} + \dfrac{x - 2}{3} - \dfrac{x - 3}{4} = 6$. *Ans.* $x = 11$.

7. $\dfrac{x}{3} - \dfrac{x}{4} - \dfrac{1}{2} = \dfrac{x}{5} - \dfrac{x}{6}$. *Ans.* $x = 10$.

8. $\dfrac{3x - 1}{7} + \dfrac{6 - x}{4} - \dfrac{2x - 4}{12} = \dfrac{54 - x}{28}$.
Ans. $x = 5$.

9. $\dfrac{5x - 7}{3} - \dfrac{3x - 2}{7} = \dfrac{x - 5}{4}$. *Ans.* $x = \dfrac{67}{83}$.

10. $\dfrac{x}{8} - \dfrac{2(x - 1)}{5} = \dfrac{3x - 4}{15} + \dfrac{x}{12}$. *Ans.* $x = \dfrac{80}{67}$.

11. $\dfrac{x - a}{3} - \dfrac{2x - 3b}{5} - \dfrac{a - x}{2} = 10a + 11b$.
Ans. $x = 25a + 24b$.

12. $\frac{6x+a}{4x+b}=\frac{3x-b}{2x-a}$. *Ans.* $x=\frac{a^2-b^2}{b-4a}$.

13. $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$.

Ans. $x=\frac{3b}{3a-2b}$.

14. $\frac{a+c}{a+x}+\frac{a-c}{a-x}=\frac{2b^2}{a^2-x^2}$ *Ans.* $x=\frac{a^2-b^2}{c}$.

15. $\frac{6x+13}{15}-\frac{3x+5}{5}=\frac{2x}{5}$. *Ans.* $x=-\frac{2}{9}$.

16. $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$. *Ans.* $x=23\frac{1}{4}$.

17. $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23$. *Ans.* $x=2$.

18. $\frac{x}{12}-\frac{8-x}{8}-\frac{5+x}{4}+\frac{11}{4}=0$.

Ans. $x=12$.

19. $3x-4-\left(\frac{4}{5}\times\frac{7x-9}{3}\right)=\frac{4}{5}\left(6+\frac{x-1}{3}\right)$.

Ans. $x=7\frac{1}{13}$.

20. $\frac{4x}{5-x}-\frac{20-4x}{x}=\frac{15}{x}$. *Ans.* $x=3\frac{2}{11}$.

21. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3}$.

Ans. $x=19$.

22. $\frac{ax}{b}+\frac{cx}{f}+g=qx+\frac{cx}{f}+h$.

Ans. $x=\frac{b(h-g)}{a-bq}$.

23. $\frac{10x+17}{18}-\frac{12x+2}{11x-8}=\frac{5x-4}{9}$. *Ans.* $x=4$.

24. $\frac{1}{7}\left(x-\frac{1}{2}\right)-\frac{1}{5}\left(\frac{2}{3}-x\right)=\frac{43}{30}.$ *Ans.* $x=\frac{43}{9}.$

25. $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}.$

Ans. $x=72.$

PROBLEMS.

81. A **problem** is a question proposed, requiring a *solution.*

82. The **solution** of a problem is the operation of finding a quantity, or quantities, that will satisfy the given conditions.

The solution of a problem consists of two parts: the *statement,* and the *solution of the equation, or equations of the problem.*

The **statement** consists in translating the conditions of the problem into algebraic language, the resulting equations being called *the equations of the problem.*

The solution of the equations is made by the general rules for solving equations.

The statement is made by representing the unknown quantities of the problem by some of the final letters of the alphabet, and then operating upon these so as to comply with the conditions of the problem. The method of stating a problem will be best learned from practical examples.

1. What number is that to which if its fifth part be added, the sum will be equal to 24?

Let x denote the number; then will $\frac{x}{5}$ denote its fifth part.

From the conditions of the problem,

$$x+\frac{x}{5}=24; \quad \text{equation of the problem.}$$

Clearing of fractions, $5x + x = 120$,
reducing, $6x = 120$,
dividing by 6. $x = 20$, the number required.

2. The sum of two numbers is 30, and their difference 6. What are the numbers?

Let x denote one number; then will $30 - x$, denote the other. From the conditions, we have,

$$(30 - x) - x = 6\text{; equation of the problem.}$$

Transposing and reducing, $-2x = -24$;
dividing by -2,

$$\left.\begin{aligned} x &= 12 \\ \therefore\ 30 - x &= 18 \end{aligned}\right\}\text{ the two numbers.}$$

3. Two couriers start from points distant 200 miles, and travel towards each other. The first travels 9 miles per hour, and the second $8\frac{1}{2}$ miles per hour. How long before they will meet, and how far will each have traveled?

Let x denote the number of hours; then will $9x$ denote the distance the first travels, and $8\frac{1}{2}x$ denote the distance the second travels. From the conditions of the problem, we have,

$$9x + 8\tfrac{1}{2}x = 200.$$

Solving,

$x = 11\frac{3}{7}$, the number of hours required.
$\therefore\ 9x = 102\frac{6}{7}$, the number of miles the first travels; and
$8\frac{1}{2}x = 97\frac{1}{7}$, the number of miles the second travels.

4. A hare starts 50 leaps before a dog, and makes 4 leaps to the dog's 3; but 2 of the dog's leaps are equal to 3 of the hare's. How many leaps must the dog make to overtake the hare?

Let x denote the number of leaps that the dog makes: then will $\frac{4x}{3}$ denote the number that the hare makes in the same time;

and if we take the length of the hare's leap as the unit of distance, the whole distance to be passed over by the dog, will be denoted by $\frac{4x}{3} + 50$: but the dog passes over a distance of $\frac{3}{2}$ units at each leap; hence, in x leaps, he will pass over a distance denoted by $\frac{3}{2}x$. From the conditions of the problem, these two quantities are equal; hence, we have,

$$\frac{4x}{3} + 50 = \frac{3x}{2}.$$

$$\therefore\ x = 300.\quad \textit{Ans.}$$

5. *A.* can do a piece of work in 9 days, and *B.* can do the same work in 10 days. In how many days can they both do it together?

Let x denote the number of days required. If we denote the work by 1, *A.* can do $\frac{1}{9}$ of it in 1 day, and in x days he can do $\frac{x}{9}$ of it. *B.* can do $\frac{1}{10}$ of it in 1 day, and in x days he can do $\frac{x}{10}$ of it. But, from the conditions, the sum of these two will be equal to the entire work; that is,

$$\frac{x}{9} + \frac{x}{10} = 1,$$

$$\therefore\ x = 4\tfrac{14}{19}\ \textit{days.}\quad \textit{Ans.}$$

6. If to a certain number its half and its eighth part be added, the sum will be equal to 78. What is the number? *Ans.* 48.

7. A man bought a horse, harness, and wagon, for 250 dollars: he gave for the harness one fourth as much as for the horse, and for the wagon as much as for the horse and harness together. What did he give for each?

Ans. \$100 for the horse, \$25 for the harness, and \$125 for the wagon.

8. A drover sold from a flock of sheep one half, and two more; he then sold half that remained, and

two more, and then he had 22 left. How many had he at first? *Ans.* 100.

9. If from 3 times a certain number we subtract 8, half the remainder will be equal to the number itself diminished by 2. What is the number? *Ans.* 4.

10. Ten years ago, a boy's age was $\frac{1}{10}$ of his father's; now, it is $\frac{1}{4}$ of it. What is the father's age now?

Ans. 60 years.

11. The sixth part of a number added to its eighth part gives 56. What is the number? *Ans.* 192.

12. Two boys had, together, 35 marbles. One fourth of the number that the first had was equal to one third of the number that the second had. How many had each? *Ans.* 20, and 15.

13. A man spent half of his money, and afterwards lost one third of what he had left, when he found that he had remaining $30. How much had he at first?

Ans. $90.

14. What number is that, from which if 5 be subtracted, one half the remainder is equal to 15? *Ans.* 35.

15. Divide $116 amongst three persons, so that the second shall have two thirds as much as the first, and the third shall have two fifths as much as the second.

Ans. $60, $40, and $16.

16. A wheat field yielded 72 bushels, which was divided between landlord and tenant in such a way, that for every five bushels that the landlord received, the tenant got seven. How many bushels did the tenant receive?

Ans. 42.

17. The half of a number exceeds its third part by 8. What is the number? *Ans.* 48.

18. A sum of money is divided between *A.*, *B.*, and *C.*, so that *A.* has $8; *B.* has as much as *A.*, together with one fifth as much as *C.*; and *C.* has as much as *A.* and *B.* together. How much has *C.*? *Ans.* $20.

19. There are 180 sheep in two flocks. If 20 are taken from the second and added to the first flock, the first flock will then contain twice as many as the second. How many sheep are there in each flock?

Ans. 100, and 80.

20. A post stands $\frac{1}{2}$ in the mud, $\frac{1}{7}$ in the water, and 10 feet in the air. What is its entire length?

Ans. 28 feet.

21. There are two numbers whose difference is 8, and the first is 5 times the second. What are the numbers?

Ans. 10, and 2.

22. A merchant gains 14 per cent. on his capital, when he finds that he has $8436. What was his capital?

Ans. $7400.

23. *A.* has 3 times as much money as *B.*; but if *A.* were to give to *B.* $100, *B.* would then have 3 times as much as *A.* How much have they each?

Ans. *A.* $150, and *B.* $50.

24. A laborer was engaged for 30 days, on condition that for every day he labored, he was to receive $2, and for every day he was idle, he was to forfeit $1. At the end of the time he received $21. How many days did he labor? *Ans.* 17.

25. *A.* is twice as old as *B.*, but 10 years ago he was three times as old. How old is *B.* now? *Ans.* 20 years.

26. Find that number which, being increased by 9, the sum divided by 2, the quotient diminished by 7, the result will be 20. *Ans.* 45.

27. Divide the number 37 into three parts, such that the first shall be 3 less than the second, and the second 5 greater than the third. *Ans.* 12, 15, and 10.

28. A man spends $\frac{2}{3}$ of his income for board, $\frac{2}{3}$ of the remainder for clothing, and has remaining $70. What is his income? *Ans.* $630.

29. Divide 1000 into two parts, so that one of them shall be $\frac{3}{5}$ of the other. *Ans.* 375, and 625.

30. A person after spending 50 dollars more than half of his income, had remaining 125 dollars more than a third of it. How much was his income? *Ans.* $1050.

31. In a naval action $\frac{1}{3}$ of a fleet was taken, $\frac{1}{6}$ of it sunk, and 2 ships burnt; $\frac{1}{7}$ of the remainder were afterwards lost in a storm, when 24 ships were left. How many ships were there in the fleet? *Ans.* 60.

32. A sum of 990 dollars was divided between *A.*, *B.*, and *C.*; *B.* received $\frac{4}{5}$ as much as *A.*; and *C.*, $\frac{5}{6}$ as much as *A.* and *B.* together. How many dollars did each receive? *Ans.* *A.*, 300; *B.*, 240; and *C.*, 450.

33. A courier *A.* starts 1165 of his own steps ahead of a courier *B.*, and takes 5 steps whilst *B.* takes but 4; now if 3 steps of *B.* are equal to 4 of the courier *A.*, how many steps must *B.* make to overtake *A.*? *Ans.* 13980.

34. The hands of a clock are together at 12 o'clock; when are they next together? *Ans.* At 1 h. $5\frac{5}{11}$ m.

35. A grazier spent $\frac{1}{12}$ of his money for horses, $\frac{1}{3}$ for oxen, and $\frac{3}{10}$ of the remainder for sheep, when he had 980 dollars left. How many dollars had he originally?

Ans. 2400.

36. Divide the number 240 into two parts, so that 7 times the first shall equal 5 times the second.

Ans. 100, and 140.

37. In a garrison of 2400 men, there are 3 times as many cavalry as artillery, and twice as many infantry as artillery and cavalry together. How many are there of each kind?

Ans. 200 artillery, 600 cavalry, and 1600 infantry.

38. Divide 21000 dollars between *A.*, *B.*, *C.*, and *D.*, so that *A.'s* part shall be $\frac{2}{3}$ of *B.'s*; *B.'s* part $\frac{4}{5}$ of *C.'s*; and *C.'s* part $\frac{6}{7}$ of *D.'s*. How many dollars will each receive?

Ans. *A.*, 3200; *B.*, 4800; *C.*, 6000; *D.*, 7000.

39. A capital was put out at $6\frac{1}{2}$ per cent for one year, when the capital and interest together amounted to 1917 dollars. How many dollars were there in the capital? *Ans.* 1800.

40. A boatman rows with the tide 42 miles in 3 hours. In returning, the tide is but $\frac{2}{3}$ as strong, and it takes $10\frac{1}{2}$ hours to row the same distance. At what rate per hour did the tide run in each case?

Ans. 6, and 4 miles.

41. A cistern can be filled by two cocks; the first would fill it in 70 minutes, and the second in 80 minutes. In how many minutes would they both fill it together? *Ans.* $37\frac{1}{3}$.

II. Equations of First Degree, containing more than one Unknown Quantity.

Explanation.

83. If we have a single equation, containing two unknown quantities, as

$$2x + 3y = 14,$$

we may find the value of one of them in terms of the other, as follows:

$$x = \frac{14 - 3y}{2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if the value of y is unknown, that of x will also be unknown; hence, from this equation alone, the value of x cannot be determined.

If now, we have a second equation,

$$3x + 2y = 11,$$

we may, in like manner, find the value of x in terms of y,

$$x = \frac{11 - 2y}{3} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the values of x and y are the same in equations (1) and (2), we shall have their second members equal to each other, giving the equation,

$$\frac{14 - 3y}{2} = \frac{11 - 2y}{3}, \quad \text{or} \quad 42 - 9y = 22 - 4y.$$

From which we find $y = 4$; and substituting this value for y, in either of the equations (1), or (2), we find $x = 1$.

Such equations are called *simultaneous.*

Simultaneous equations are those in which the values of the unknown quantities are the same in both.

We have seen that it requires two simultaneous equations, containing two unknown quantities, to determine the values of the unknown quantities. In the same way, it could be shown that it would require three equations containing three unknown quantities, four equations containing four unknown quantities, and so on, to determine the values of the unknown quantities. In general, there must be as many equations as there are unknown quantities. The equations necessary to determine any number of unknown quantities, constitute a *group of simultaneous equations.*

Such equations are solved by *successive elimination.*

ELIMINATION.

84. **Elimination** is the operation of combining two equations so as to get rid of one of the unknown quantities which enter them.

There are three principal methods of elimination: *by addition, or subtraction; by substitution;* and *by comparison.*

1°. Elimination by Addition, or Subtraction.

85. Take the equations,

$$7x + 6y = 20 \quad . \quad . \qquad (1)$$

$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Multiplying both members of (1) by 4, and of (2) by 6, (axiom 3°), we have,

$$28x + 24y = 80 \quad . \; . \; . \; . \; . \quad (3)$$
$$54x - 24y = 84 \quad . \; . \; . \; . \; . \quad (4)$$

Adding (3) and (4), member to member, (axiom 1°), we have,

$$82x = 164.$$

Here, y has been eliminated *by addition.*

Again, multiplying both members of (1) by 9, and of (2) by 7,

$$63x + 54y = 180 \quad . \; . \; . \; . \quad (5)$$
$$63x - 28y = 98 \quad . \; . \; . \; . \quad (6)$$

Subtracting (6) from (5), member from member, we have,

$$82y = 82.$$

Here, x has been eliminated *by subtraction.* In the same manner, an unknown quantity may be eliminated from any two simultaneous equations; hence, the

RULE.

Prepare the equations, so that the coefficients of the quantity to be eliminated shall be numerically equal in both; if their signs are unlike, add the equations, member to member; if alike, subtract them, member from member,

In *preparing* the above equations, we multiplied both members of each, by the coefficient of the quantity to be eliminated in the other. They may be prepared in other ways. A better way, in most cases, is to find the least common multiple of the coefficients of the quantity to be eliminated; then multiply both members of each equation by the quotient of this least common multiple by the coefficient of the quantity to be eliminated in that equation. In the first case considered, the least common multiple of 4 and 6 is 12; we might have multiplied both members of (1) by $\frac{12}{6}$, or 2, and of (2) by $\frac{12}{4}$, or 3, giving,

$$14x + 12y = 40$$
$$27x - 12y = 42.$$

Whence, by addition,

$$41x = 82.$$

Here, y has been eliminated as before, but we have a simpler equation.

2°. Elimination by Substitution.

86. Take the same equations as before:

$$7x + 6y = 20 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Finding, from (1), the value of y in terms of x,

$$y = \frac{20 - 7x}{6}.$$

Substituting this value of y, in (2), we have,

$$9x - 4\left(\frac{20 - 7x}{6}\right) = 14.$$

Here, y has been eliminated *by substitution.* In the same manner, we may eliminate an unknown quantity between any two simultaneous equations; hence, the

RULE.

Find from one of the equations the value of the quantity to be eliminated, in terms of the *other quantities; substitute this value for that quantity in the other equation.*

3°. Elimination by Comparison.

87. Take the same equations as before:

$$7x + 6y = 20 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$9x - 4y = 14 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Finding the values of x in terms of y, from each of equations (1) and (2), we have,

$$x = \frac{20 - 6y}{7};$$

$$x = \frac{14 + 4y}{9}.$$

Placing these two values equal to each other, we have,

$$\frac{20 - 6y}{7} = \frac{14 + 4y}{9}.$$

Here, x has been eliminated *by comparison.* In the same manner, an unknown quantity may be eliminated between any two simultaneous equations; hence, the

RULE.

Find from each equation the value of the quantity to be eliminated; place these values equal to each other.

Of the three rules given, any one can be used, as may be most convenient. As a general thing, that is employed which gives rise to the simplest equations.

SOLUTION OF GROUPS OF SIMULTANEOUS EQUATIONS.

88. Take the group of three equations:

$$3x + 4y - 2z = 10 \quad . \quad . \quad . \quad (1)$$
$$5x - 2y + 3z = 16 \quad . \quad . \quad . \quad (2)$$
$$4x + 2y + 2z = 22 \quad . \quad . \quad . \quad (3)$$

Combining (1) and (2), also (1) and (3), eliminating z in each case, we have the new group,

$$19x + 8y = 62 \quad . \quad . \quad . \quad (4)$$
$$7x + 6y = 32 \quad . \quad . \quad . \quad (5)$$

Combining (4) and (5), eliminating y, we have the single equation,

$$29x = 58; \qquad \therefore\ x = 2.$$

Substituting this value of x in (5), we have,

$$14 + 6y = 32; \qquad \therefore\ y = 3.$$

Substituting these values of x and y in (1), we have,

$$6 + 12 - 2z = 10; \qquad \therefore\ z = 4.$$

In the same manner, any group of simultaneous equations may be solved; hence, the

RULE.

I. Combine one equation of the group with each of the others, eliminating one unknown quantity: there will result a new group containing one equation less than the original group.

II. Combine one equation of this group with each of the others, eliminating a second unknown *quantity: there will result a new group containing two equations less than the original group.*

III. Continue the operation until a single equation is found, containing but one unknown quantity.

IV. Find the value of this unknown quantity by the preceding rules; substitute this in either one of the group of two equations, and find the value of a second unknown quantity; substitute these in any one of the group of three, finding a third unknown quantity; and so on, till the values of all are found.

In making the combinations, care should be taken to make them in such a way as to obtain as simple equations as possible. If any unknown quantity does not enter all of the equations, it will generally be best to eliminate that quantity first.

EXAMPLES.

1. Solve the equations,

$$\frac{x}{3} + \frac{y}{5} = 5 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$2x + \frac{y}{3} = 17 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Clearing of fractions,

$$5x + 3y = 75 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$6x + y = 51 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Combining (3) and (4),

$$13x = 78; \quad \therefore x = 6.$$

Substituting in (4),

$$36 + y = 51; \quad \therefore y = 15.$$

2. Solve the equations,

$$3x - 4y + 5z = 14 \quad . \quad . \quad . \quad . \quad . \tag{1}$$
$$3y + 2z = 10 \quad . \quad . \quad . \quad . \quad . \tag{2}$$
$$12x - 8y - z = 30 \quad . \quad . \quad . \quad . \quad . \tag{3}$$

Combining (1) and (3),

$$8y - 21z = -26 \quad . \quad . \quad . \quad . \quad . \tag{4}$$

Combining (2) and (4),

$$79z = 158; \qquad \therefore\ z = 2.$$

By successive substitution, we have,

$$y = 2, \quad \text{and} \quad x = 4.$$

Solve the following groups of simultaneous equations:

3. $$\left.\begin{aligned} 3x + 4y &= 18 \\ 2x - y &= 1 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= 2. \\ y &= 3. \end{aligned}\right.$$

4. $$\left.\begin{aligned} 7x - 3y &= 12 \\ 2x + 2y &= 12 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= 3. \\ y &= 3. \end{aligned}\right.$$

5. $$\left.\begin{aligned} 5x + 3y &= 26 \\ 5x - y &= 18 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= 4. \\ y &= 2. \end{aligned}\right.$$

6. $$\left.\begin{aligned} 4x + 3y &= 16 \\ 3x + 4y &= 19 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= 1. \\ y &= 4. \end{aligned}\right.$$

7. $$\left.\begin{aligned} 6x + y &= 12 \\ x + 6y &= 37 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= 1. \\ y &= 6. \end{aligned}\right.$$

8. $$\left.\begin{aligned} 4x + 5y &= 17 \\ 3y - 2x &= 8 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= \tfrac{1}{2}. \\ y &= 3. \end{aligned}\right.$$

9. $$\left.\begin{aligned} x - 3y &= 6 \\ 2x + 9y &= 17 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x &= 7. \\ y &= \tfrac{1}{3}. \end{aligned}\right.$$

10. $$\left.\begin{array}{r} x + y + z = 6 \\ 5x + 2y - 3z = 0 \\ 2x + y - z = 1 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 1. \\ y = 2. \\ z = 3. \end{array}\right.$$

11. $$\left.\begin{array}{r} 2x - \frac{3}{4}y = 9 \\ x + y = 21 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 9. \\ y = 12. \end{array}\right.$$

12. $$\left.\begin{array}{r} 3x + 2y - z = 7 \\ x + y - z = 1 \\ x + 2y + 3z = 15 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 2. \\ y = 2. \\ z = 3. \end{array}\right.$$

13. $$\left.\begin{array}{r} \frac{x}{2} - y = 1 \\ x - \frac{y}{2} = 8 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 10. \\ y = 4. \end{array}\right.$$

14. $$\left.\begin{array}{r} \frac{x + y}{10} + \frac{x - y}{2} = 0 \\ \frac{x + y}{5} + \frac{x - y}{2} = 1 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 4. \\ y = 6. \end{array}\right.$$

15. $$\left.\begin{array}{l} \frac{2x - y}{4} - \frac{3}{2} = \frac{3y}{4} - x - 2 \\ \frac{x + y}{3} = 2\frac{2}{3} \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 3. \\ y = 5. \end{array}\right.$$

16. $$\left.\begin{array}{r} \frac{x}{2} + \frac{y}{3} = 12 \\ \frac{x}{3} + \frac{y}{2} = 13 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 12. \\ y = 18. \end{array}\right.$$

17. $$\left.\begin{array}{r} 2x - 2y + 3z = 16 \\ 3x + 5y - 2z = 6 \\ 4x + 3y - 4z = -1 \end{array}\right\} \qquad Ans. \left\{\begin{array}{l} x = 3. \\ y = 1. \\ z = 4. \end{array}\right.$$

18. $$\left.\begin{aligned} \frac{x}{3}+\frac{y}{4}+\frac{z}{5}&=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}&=38 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=62 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=24. \\ y&=60. \\ z&=120. \end{aligned}\right.$$

19. $$\left.\begin{aligned} \frac{x+y}{z}&=5 \\ \frac{y-z}{x}&=1 \\ \frac{x-2}{y}&=\tfrac{1}{3} \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=4. \\ y&=6. \\ z&=2. \end{aligned}\right.$$

20. $$\left.\begin{aligned} 3.4x-.02y&=.01 \\ 2x+.\,4y&=1.2 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=.02. \\ y&=2.9. \end{aligned}\right.$$

21. $$\left.\begin{aligned} \frac{2x+y}{9}+\frac{7x+6y+11}{18}&=\frac{68-4x}{6} \\ \frac{21}{20}\left(\frac{x}{7}+\frac{y}{4}+\frac{4}{3}\right)&=4x-\frac{y}{8}-24 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=7. \\ y&=4. \end{aligned}\right.$$

22. $$\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}&=1-\frac{x}{c} \\ \frac{y}{a}+\frac{x}{b}&=1+\frac{y}{c} \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=\frac{(ab+ac-bc)abc}{a^2b^2+a^2c^2-b^2c^2}. \\ y&=\frac{(ac-ab-bc)abc}{a^2b^2+a^2c^2-b^2c^2}. \end{aligned}\right.$$

23. $$\left.\begin{aligned} x+\tfrac{1}{2}(y+z)&=102 \\ y+\tfrac{1}{3}(x+z)&=78 \\ z+\tfrac{1}{4}(x+y)&=61 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=62. \\ y&=46. \\ z&=34. \end{aligned}\right.$$

24. $$\left.\begin{aligned} \frac{6y-4x}{3z-7}&=1 \\ \frac{5z-x}{2y-3z}&=1 \\ \frac{y-2z}{3y-2x}&=1 \end{aligned}\right\} \qquad \textit{Ans.} \left\{\begin{aligned} x&=10. \\ y&=7. \\ z&=3. \end{aligned}\right.$$

25. $$\left.\begin{aligned}\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}&=\frac{38}{5}\\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}&=\frac{61}{6}\\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}&=\frac{161}{10}\end{aligned}\right\}\qquad \textit{Ans.}\ \left\{\begin{aligned}x&=\frac{1}{2}.\\ y&=\frac{1}{3}.\\ z&=\frac{1}{4}.\end{aligned}\right.$$

Example 25 may be solved by assuming

$$x'=\frac{1}{x},\quad y'=\frac{1}{y},\quad \text{and}\quad z'=\frac{1}{z},$$

which gives for the group 25.

$$\left.\begin{aligned}3x'-\frac{4y'}{5}+\ z'&=\frac{38}{5}\\ \frac{x'}{3}+\frac{y'}{2}+2z'&=\frac{61}{6}\\ \frac{4x'}{5}-\frac{y'}{2}+4z'&=\frac{161}{10}\end{aligned}\right\}\quad\therefore\quad\left\{\begin{aligned}x'&=2.\\ y'&=3.\\ z'&=4.\end{aligned}\right.$$

From which the values of x, y, and z, may readily be found.

26. $$\left.\begin{aligned}\frac{2}{x}-\frac{5}{3y}+\frac{1}{z}&=\frac{85}{27}\\ \frac{1}{4x}+\frac{1}{y}+\frac{2}{z}&=\frac{443}{72}\\ \frac{5}{6x}-\frac{1}{y}+\frac{4}{z}&=\frac{433}{36}\end{aligned}\right\}\qquad \textit{Ans.}\ \left\{\begin{aligned}x&=6.\\ y&=9.\\ z&=\frac{1}{3}.\end{aligned}\right.$$

PROBLEMS.

1. Find two numbers whose sum is a, and whose difference is b.

Let x denote the greater number, and y the less number.

From the conditions of the problem, we have,

$$x + y = a$$
$$x - y = b$$

which are equations of the problem.

Solving, by preceding rules, we have,

$$x = \frac{a}{2} + \frac{b}{2}, \text{ and } y = \frac{a}{2} - \frac{b}{2}; \textit{ Ans.}$$

Since a and b may be any numbers whatever, we have these following principles by means of which all similar cases can be solved:

1°. *The greater number is equal to the half sum of the two numbers increased by the half difference.*

2°. *The less number is equal to the half sum of the two numbers diminished by the half difference.*

2. If 2 is added to the numerator of a certain fraction, its value will become $\frac{3}{4}$; but if 2 is added to the denominator, its value will be $\frac{1}{2}$. What is the fraction?

Let x denote the numerator, and y the denominator.

From the given conditions, we have the equations of the problem,

$$\frac{x+2}{y} = \frac{3}{4}, \text{ and } \frac{x}{y+2} = \frac{1}{2};$$

whence, $x = 7$, and $y = 12$; hence, the fraction is $\frac{7}{12}$.

3. The hands of a clock are together at 12 o'clock; when are they next together?

Let x denote the number of minute spaces passed over by the minute hand, and y the number of minute spaces passed over by the hour hand.

From the nature of the problem, we have,

$$x = y + 60,$$
$$x = 12y.$$
$$\therefore \quad x = 65\tfrac{5}{11}, \qquad y = 5\tfrac{5}{11}.$$

Hence, they are together at 1 h. $5\frac{5}{11}$ m.

This problem has already been solved by means of a single unknown quantity; many of the following problems can also be solved in the same manner.

4. A person has 22000 dollars at interest, which yields him 1220 dollars annually; a part bears interest at 5 per cent., and the remainder at 6 per cent. How many dollars in each part?

Let x denote the number of dollars in the first part, and y the number of dollars in the second part.

From the conditions of the problem, we have,

$$x + y = 22000$$
$$x \times \tfrac{5}{100} + y \times \tfrac{6}{100} = 1220.$$
$$\therefore \quad x = 10000, \qquad y = 12000.$$

5. *A.'s* age is equal to twice *B.'s* age; 20 years ago, *A.'s* age was 4 times *B.'s* age. What are their ages?

Ans. *A.'s* 60; *B.'s* 30.

6. There are two numbers: the first added to half the second gives 35; the second added to half the first gives 40. What are the numbers? *Ans.* 20 and 30.

7. A man has three sons: the sum of the ages of the first and second is 27, that of the first and third is 29, and that of the second and third is 32. What are the ages of each? *Ans.* 12, 15, and 17.

8. Two men are in trade; the stock of the first increased by one third that of the second, is $1700;

the stock of the second increased by one fourth that of the first, is $1800. What is the stock of each?

Ans. $1200 and $1500.

9. Find two numbers such that $\frac{1}{2}$ the first plus $\frac{1}{3}$ the second shall equal 45, and $\frac{1}{2}$ the second plus $\frac{1}{5}$ of the first shall equal 40. *Ans.* 50 and 60.

10. The sum of the first and second of three numbers is 13, that of the first and third 16, and that of the second and third 19. What are the numbers?

Ans. 5, 8, and 11.

11. Bought 100 lbs. of sugar and 80 lbs. of coffee for $28, and afterwards bought at the same rates 200 lbs. of sugar and 60 lbs. of coffee for $36. What did each cost per pound?

Ans. Sugar 12 cents, and coffee 20 cents.

12. There are three numbers; the first increased by twice the second and three times the third, makes 74; the second, increased by twice the third and three times the first, makes 90; the third, increased by twice the first and three times the second, makes 100. What are the numbers? *Ans.* 20, 18, and 6.

13. A butcher bought of one person 12 sheep and 20 lambs for 44 dollars, and of a second person 7 sheep and 13 lambs for 27 dollars, at the same rates. How many dollars did he give apiece?

Ans. $2 for sheep, and $1 for lambs.

14. Divide the number 1152 into three parts, such that 9 times the sum of the first and second shall be

equal to 7 times the sum of the second and third; and if 8 times the first be subtracted from 8 times the second, the remainder shall be equal to the sum of the first and third. *Ans.* 288, 384, and 480.

15. A farmer mixed rye and oats, forming 100 bushels of the mixture. The rye was worth 96 cents per bushel, the oats 56 cents, and the mixture 72 cents. How many bushels did he use of each?

Ans. 40 of rye, and 60 of oats.

16. A person has two sorts of wine, one worth 40 cents a quart, and the other 24 cents. How much of each kind must he use to form a gallon worth 112 cents?

Ans. 1 quart of the first, 3 quarts of the second.

17. *A.*, and *B.*, trade on a joint stock of 833 dollars, and clear 153 dollars. *A.'s* share of the gain is 45 dollars more than *B.'s*. What share of the capital did each possess? *Ans.* *A.*, $539; *B.*, $294.

18. Two laborers, *A.*, and *B.*, received 51 dollars. *A.* had been employed 14 days, and *B.* 15 days; *A.* received for 6 days' labor 1 dollar more than *B.* got for 4 days' labor. How many dollars did each receive per day? *Ans.* *A.*, $1\frac{1}{2}$; *B.*, 2.

19. In 80 pounds of an alloy of copper and tin, there are 7 lbs. of copper to 3 of tin. How much copper must be added to the alloy, that there may be 11 lbs. of copper to 4 of tin? *Ans.*. 10 lbs.

20. In 3 battalions there are 1905 troops; $\frac{1}{2}$ the number in the first, together with $\frac{1}{3}$ the number in the

second, is 60 less than the number in the third; $\frac{1}{2}$ the number in the third, together with $\frac{1}{3}$ the number in the first, is 165 less than the number in the second. How many are there in each battalion?

Ans. 630, 675, and 600.

21. A grocer has three kinds of tea: 12 lbs. of the first, 13 of the second, and 14 of the third are together worth 25 dollars: 10 of the first, 17 of the second, and 11 of the third are together worth 24 dollars; 6 of the first, 12 of the second, and 6 of the third are together worth 15 dollars. What is the value of a pound of each?

Ans. 50 cents, 60 cents, and 80 cents.

22. *A.* owes $1200, and *B.* $2500; but neither has money enough to pay his debts. Says *A.* to *B.*, "lend me $\frac{1}{8}$ of your fortune, and I can pay my debts;" says *B.* to *A.*, "lend me $\frac{1}{9}$ of your fortune, and I can pay mine." What fortune had each?

Ans. *B.*, had $2400; and *A.*, $900.

23. The united ages of a father and son are 80 years; and if the age of the son be doubled, it will exceed the father's age by 10 years. What is the age of each?

Ans. 50, and 30.

24. *A.* travels uniformly along a certain road, *B.* starts an hour afterwards in pursuit, and after 4 hours finds by inquiry that he is travelling $1\frac{1}{2}$ miles per hour slower than *A.*; he then doubles his rate of travel, and overtakes *A.*, $6\frac{1}{3}$ hours from the time he started in pur-

suit. At what rate did *A*. travel, and what was the rate that *B*. traveled at first?

Ans. *A.'s* rate, $9\frac{3}{4}$ miles; *B.'s*, $8\frac{1}{4}$.

25. There are 32 gallons of wine in two casks. If from the first there be drawn into the second as much as there is in the second; then if there be drawn from the second into the first as much as remains in the first; and then if there be drawn from the first into the second as much as remains in the second, there will be 16 gallons in each cask. How many gallons were there originally in each? *Ans.* 22, and 10.

26. A cistern can be filled by 3 pipes. The first can fill it in 4 hours, the first and second together can fill it in 3 hours, and the third can fill it in 2 hours. How long will it take for them all to fill it together, and how long will it take the second alone to fill it?

Ans. All in 1 h. 12 m.; the second in 12 hours.

27. A cistern has two discharge cocks: they both run together for two hours when the first one is closed; the second one then empties it in 2 hours and 48 minutes. Had the second one been closed at the end of two hours, the first one would have emptied it in 4 hours and 40 minutes. In what time could each empty it alone?

Ans. The first in 10 hours; the second in 6 hours.

III. EXPLANATION OF SYMBOLS AND DISCUSSION OF PROBLEMS.

Explanations and Principles.

89. The symbol 0 is called *zero;* the symbol ∞ is called *infinity;* and the symbol $\frac{0}{0}$ is called *the symbol of indetermination.*

To explain the meaning of these symbols, let us take the equation,

$$t = \frac{a}{d} \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

Which can be written under the form

$$d \times t = a \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Any set of values of a, d, and t, that will satisfy equation (2) will, of necessity, satisfy equation (1).

1°. If we suppose a to be equal to 0, and d to be *finite,* that is, to contain a limited number of units, equation (2) will become

$$d \times t = 0.$$

It is obvious that 0 is the only value of t that will satisfy this equation. Making $a = 0$ and $t = 0$ in (2) and (1), we have,

$$0 \times d = 0; \quad \text{and} \quad \frac{0}{d} = 0.$$

Hence, we say that 0 *multiplied, or divided, by a finite quantity is equal to* 0.

2°. If we suppose d to be equal to 0, and a to be finite, equation (2) becomes

$$0 \times t = a.$$

It is obvious that no finite value of t can satisfy the last equation; this fact we express by saying that t is *infinite*, that is, that it is greater than any assignable quantity. Making $d = 0$ and $t = \infty$ in equation (1), we have,

$$\frac{a}{0} = \infty, \text{ whence } \frac{a}{\infty} = 0.$$

Hence we say that *a finite quantity divided by* 0 *is equal to infinity*, and that *a finite quantity divided by infinity is equal to* 0.

3°. If both a and d are supposed equal to 0, equation (2) becomes

$$0 \times t = 0.$$

It is obvious that this equation will be satisfied for every finite value of t, (principle 1°). Making $a = 0$ and $d = 0$ in (1), we have

$$t = \frac{0}{0}.$$

Which is true for all finite values of t. In this case the equation does not determine the value of t, a fact that we express by saying that t is *indeterminate*. Hence, we say that *an indeterminate quantity is one that has an infinite number of values.*

A fraction may reduce to the indeterminate form in consequence of a common factor in both terms, which factor becomes 0, for the particular hypothesis. Thus, the fraction

$$\frac{7(x+1)(x-1)}{2(x-1)}$$

reduces to $\frac{0}{0}$ for the particular value $x = 1$. If we strike out the common factor $x - 1$ and then make $x = 1$, we find the true value of the fraction to be 7. Before deciding on the nature of the expression $\frac{0}{0}$, we must, therefore, determine whether it results from the existence of a common factor which reduces to 0 for the particular hypothesis; if not it is a true symbol of indetermination.

Definitions.

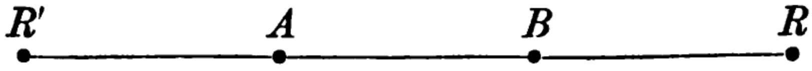
90. The **discussion** of a problem consists in making every possible supposition on the *arbitrary* quantities which enter it, and interpreting the results.

An **arbitrary quantity** is a quantity to which a value may be given at pleasure.

The method of interpreting results is illustrated in the solution and discussion of the following problem:

Problem of the Couriers.

91. Two couriers, *A.* and *B.*, travel along the same line, *R′ R*, in the same direction, *R′* towards *R*, and at uniform rates; the courier *A.* travels *m* miles per hour, and the courier *B.*, *n* miles per hour. Now, supposing them to be separated by a distance *a* at any *epoch*, say 12 o'clock, when are they together?



Let the position of the rearmost courier, *A.*, be taken as the origin of distances, and suppose all distances estimated towards *B.* to be positive.

Denote the number of hours from the epoch to the time they are together by t. Denote the distance the courier B. travels in the time t, by x; then will the distance that the courier A. travels, in the same time, be denoted by $a + x$.

Then, since the distance traveled is equal to the number of hours multiplied by the rate per hour, we have the equations:

$$mt = a + x,$$
$$nt = x;$$

whence, by solving,

$$t = \frac{a}{m - n}.$$

Discussion.

92. In discussing the value of t, found in the last article, it is to be observed that the distance between the couriers may be assumed at pleasure; hence, a is arbitrary: the rates of travel may also be assumed at pleasure; hence, m and n are arbitrary.

From the conditions of the problem, a can never be negative; hence, the only suppositions that can be made on a are $a > 0$, and $a = 0$. The only suppositions that can be made on m and n are $m > n$, $m < n$, and $m = n$. By combining these hypotheses we obtain six suppositions, as follows:

1°. $a > 0$, and $m > n$. 2°. $a > 0$, and $m < n$.
3°. $a = 0$, and $m > n$. 4°. $a = 0$, and $m < n$.
5 °. $a > 0$, and $m = n$. 6°. $a = 0$, and $m = n$.

We shall consider these hypotheses in order:

1°. $a > 0$, and $m > n$; 2°. $a > 0$, and $m < n$.

The first supposition makes the value of t, in article 91, essentially *positive*, and the second supposition makes the value of t essentially *negative*.

In the first case, *A.* is behind *B.* at the epoch, 12 o'clock, and is continually gaining on him; hence, *A.* will overtake *B.* at some time after 12 o'clock, and the couriers will be together. We therefore interpret the *positive* value of t as indicating that the time when they are together is *after* 12 o'clock.

In the second case, *A.* is behind *B.* at 12 o'clock, and *B.* is continually gaining on *A.*; hence, they can never be together after 12 o'clock: it is plain, however, that they must have been together at some time before 12 o'clock. We therefore interpret the *negative* value of t as indicating that the time when they are together is *before* 12 o'clock.

These results conform to the principle of interpreting positive and negative quantities as explained in Article 6.

3°. $a = 0$, and $m > n$; 4°. $a = 0$, and $m < n$.

The third supposition makes the value of t equal to $+ 0$, and the fourth supposition makes it equal to $- 0$.

In both cases the couriers are together at 12 o'clock, and since they travel at unequal rates, it is obvious that they can never be together after 12 o'clock, nor can they have been together at any time before 12 o'clock. We therefore interpret the results $+ 0$, and $- 0$, as indicating that *no time* is to be added to, or subtracted from 12 o'clock, to find the time when they are together; that is, they are together at 12 o'clock, and at no other time.

5°. $a > 0$, and $m = n$.

The fifth supposition makes the value of t equal to a divided by 0, or equal to ∞.

In this case the couriers are separated by the distance a at 12 o'clock, and since they travel equally fast it is obvious that they always have been, and always will be, separated by that interval. We therefore interpret the result ∞, as indicating that the interval from 12 o'clock till the time they are together, is greater than any assignable time, that is, that they are *never* together.

6°. $a = 0$, and $m = n$.

The sixth supposition makes the value of t equal to $\frac{0}{0}$. In this case the couriers are together at 12 o'clock, and since they travel equally fast they have always been, and always will be, together. We therefore interpret the result $\frac{0}{0}$, as indicating that there are an infinite number of times, both before and after 12 o'clock, when they are together, that is, they are always together.

From what precedes, we see that 0, ∞, and $\frac{0}{0}$, though not quantities, are nevertheless symbols which, if properly interpreted, indicate correct answers to problems.

Extension of the Formula of Article 91.

93. The formula, $t = \frac{a}{m - n}$, may be used in solving other questions similar to the problem of the couriers.

If we call a, the initial distance, and $m - n$, the relative rate of travel, the formula may be expressed by saying, the time elapsed is equal to the *initial distance* divided by the *relative velocity.*

As an example, let it be required to find when the hands of a clock are together between 1 and 2 o'clock: here, 12 o'clock is taken as the origin of distance; if we take the minute space on the dial as the unit, the *initial distance,* that is, the distance to be *gained,* is 60; the rate of the minute hand is 60, that of the hour hand, 5: hence,

$$t = \frac{60}{60 - 5} = 1\tfrac{1}{11} \text{ hours, or } 1 \text{ h. } 5\tfrac{5}{11} \text{ min.}$$

To find when the hands are together between 2 and 3 o'clock, we have the initial space, 2 × 60, or 120, and the rates as before. Hence,

$$t = \frac{120}{60 - 5} = 2\tfrac{2}{11} \text{ hours, or } 2 \text{ h. } 10\tfrac{10}{11} \text{ min.}$$

CHAPTER VI.

FORMATION OF POWERS.

I. POWERS OF MONOMIALS.

Definitions.

94. A **power** is the product of two or more equal factors; one of these factors is called the **root** of the power.

The product of *two* equal factors is called **a second power,** or **a square**; the product of *three* equal factors is called **a third power,** or **a cube**; the product of *four* equal factors is called **a fourth power**; and so on.

The *degree* of a power is indicated by its exponent. Thus, a^4 denotes the *fourth* power of a; and a^n denotes the n^{th} power of a.

The root is called the first power, and by analogy a quantity written with a negative, or with a fractional exponent is also called a power. Thus, a, denotes the *first* power of a; a^{-3}, denotes the *minus* 3 power of a; $a^{\frac{2}{5}}$, denotes the $\frac{2}{5}$*th* power of a; $a^{-\frac{3}{4}}$, denotes the $-\frac{3}{4}$*th* power of a.

Demonstration of Rule.

95. Let it be required to find the *third* power of $7a^2x$: from the definition of a power and the rule for multiplication, we have,

$$(7a^2x)^3 = 7a^2x \times 7a^2x \times 7a^2x = 343a^6x^3.$$

In like manner any monomial may be raised to any power; hence, the following rule for raising a monomial to any power:

RULE.

Raise the coefficient to the required power for a new coefficient; write after this all the letters, giving to each an exponent equal to the product of its original exponent by the exponent of the power.

If the given monomial is positive, all of its powers are positive; if it is negative, its square is positive, its cube negative, its fourth power positive, and so on. In general, even powers of a negative quantity are positive, and odd powers negative. These principles follow from the *rule for signs*, in multiplication.

EXAMPLES.

1. $(3ax^2y)^2$. *Ans.* $9a^2x^4y^2$.
2. $(2a^2yx^3)^3$. *Ans.* $8a^6y^3x^9$.
3. $(-2axy^2)^3$. *Ans.* $-8a^3x^3y^6$.
4. $(-3a^2bc^3x)^4$. *Ans.* $81a^8b^4c^{12}x^4$.
5. $(-7dx^3y^2)^3$. *Ans.* $-343d^3x^9y^6$.
6. $(2x^3yz)^5$. *Ans.* $32x^{15}y^5z^5$.
7. $(-d^2x^3y^4)^3$. *Ans.* $-d^6x^9y^{12}$.

8. $(-x^2y^3z^4)^4$. *Ans.* $x^8y^{12}z^{16}$.

9. $(4axy^3z)^3$. *Ans.* $64a^3x^3y^9z^3$.

10. $(-3a^2y^3)^4$. *Ans.* $81a^8y^{12}$.

Powers of Fractions.

96. Let it be required to find the *third* power of $\frac{2a^2x}{3by}$:

From the definition of a power, and the rule for the multiplication of fractions, we have,

$$\left(\frac{2a^2x}{3by}\right)^3 = \frac{2a^2x}{3by} \times \frac{2a^2x}{3by} \times \frac{2a^2x}{3by} = \frac{8a^6x^3}{27b^3y^3};$$

and similarly for other fractions; hence, the

RULE.

Raise the numerator to the required power for a new numerator, and the denominator to the required power for a new denominator.

The rule for signs is the same as in the last article.

EXAMPLES.

1. $\left(\frac{a}{b}\right)^2$. *Ans.* $\frac{a^2}{b^2}$.

2. $\left(\frac{ax}{by}\right)^3$ *Ans.* $\frac{a^3x^3}{b^3y^3}$.

3. $\left(\frac{-2ax}{3y}\right)^4$. *Ans.* $\frac{16a^4x^4}{81y^4}$.

4. $\left(\frac{-3y}{4x}\right)^3$. *Ans.* $-\frac{27y^3}{64x^3}$.

5. $\left(\frac{2ax^2y}{3bc^2}\right)^2$. *Ans.* $\frac{4a^2x^4y^2}{9b^2c^4}$.

6. $\left(-\frac{dx}{3y^2}\right)^3$. *Ans.* $-\frac{d^3x^3}{27y^6}$.

7. $\left(\frac{axy^3}{2bz^2}\right)^3$. *Ans.* $\frac{a^3x^3y^9}{8b^3z^6}$.

8. $\left(-\frac{3ay^4}{2b^2x}\right)^4$. *Ans.* $\frac{81a^4y^{16}}{16b^8x^4}$.

9. $\left(-\frac{2x^2y^3}{abc}\right)^2$. *Ans.* $\frac{4x^4y^6}{a^2b^2c^2}$.

10. $\left(-\frac{2ab^2c^3x^4}{3}\right)^6$. *Ans.* $\frac{64a^6b^{12}c^{18}x^{24}}{729}$.

Extension of the Preceding Rules.

97. The rule for raising a monomial to any power holds true when the exponents of any of the letters are negative, or when the exponent of the required power is negative.

Let it be required to find the square of $3a^{-2}x^{-4}$, and that power of $2ax^2$, whose exponent is -3: it has been shown that any factor may be changed from the denominator to the numerator, or from the numerator to the denominator, by changing the sign of its exponent (Art. 32); hence,

$$(3a^{-2}x^{-4})^2 = \left(\frac{3}{a^2x^4}\right)^2 = \frac{9}{a^4x^8};$$

also,
$$(2ax^2)^{-3} = \frac{1}{(2ax^2)^3} = \frac{1}{2^3a^3x^6}.$$

Transferring factors to the numerator, we have,

$$(3a^{-2}x^{-4})^2 = 9a^{-4}x^{-8};$$

$$\text{also,} \quad (2ax^2)^{-3} = 2^{-3}a^{-3}x^{-6} = \tfrac{1}{8}a^{-3}x^{-6},$$

which results conform to preceding rules.

EXAMPLES.

1. $(a^{-2})^2$. *Ans.* a^{-4}.
2. $(x^{-3}y)^{-2}$. *Ans.* x^6y^{-2}.
3. $(2x^2y^3)^{-2}$. *Ans.* $\frac{1}{4}x^{-4}y^{-6}$.
4. $(2x^{-2}y^{-3})^{-2}$. *Ans.* $\frac{1}{4}x^4y^6$.
5. $(ax^2y^3z^{-2})^{-3}$. *Ans.* $a^{-3}x^{-6}y^{-9}z^6$.
6. $(2a^{-3}b^{-2}c^3)^3$. *Ans.* $8a^{-9}b^{-6}c^9$.
7. $(-3x^{-1}y^{-2})^3$. *Ans.* $-27x^{-3}y^{-6}$.
8. $(5a^{-2}b^{-3}c^{-2})^{-3}$. *Ans.* $\frac{1}{125}a^6b^9c^6$.
9. $(-2x^2y^{-3})^{-3}$. *Ans.* $-\frac{1}{8}x^{-6}y^9$.
10. $(-2x^{-2}y^{-3})^4$. *Ans.* $16x^{-8}y^{-12}$.
11. $(-3ax^2y^{-1}z^{-2})^{-2}$. *Ans.* $\frac{1}{9}a^{-2}x^{-4}y^2z^4$.
12. $(-3x^{-3}y^4)^{-4}$. *Ans.* $\frac{1}{81}x^{12}y^{-16}$.

II. POWERS OF BINOMIALS. BINOMIAL FORMULA.

Explanation.

98. A binomial may be raised to any power by the process of continued multiplication, but when the exponent of the power is greater than 2, the operation is greatly abridged by making use of the *binomial formula.*

Definitions.

99. The **binomial formula**, is a formula by means of which a binomial may be raised to any power, without going through the process of continued multiplication.

Demonstration.

100. The following powers of $x + y$ are found by actual multiplication:

$$\begin{aligned}
(x + y)^1 &= x + y. \\
(x + y)^2 &= x^2 + 2xy + y^2. \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3. \\
(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4. \\
(x + y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.
\end{aligned}$$

And in the same way, the higher powers might be obtained. If we examine the powers already deduced, we see that they are all formed according to the following laws:

1°. LAW OF EXPONENTS.—*The exponent of the leading letter in the first term is equal to the exponent of the power, and the exponent of that letter goes on diminishing by 1 in each term towards the right till the last term, where it is 0: the exponent of the following letter is 0 in the first term, and the exponent of that letter goes on increasing by 1 in each term towards the right to the last term, where it is equal to the exponent of the power.*

2°. Law of Coefficients. — *The coefficient of the first term is 1; the coefficient of any succeeding term is found by multiplying the coefficient of the preceding term by the exponent of the leading letter in that term, and dividing the product by the number of terms preceding the required term.*

Let us assume that these laws of formation hold true for a power whose exponent is m, m being any positive whole number. The application of these laws gives,

$$(x + y)^m = x^m + mx^{m-1}y + m\,.\,\frac{m-1}{2}x^{m-2}y^2$$
$$+ m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}x^{m-3}y^3 + \&c., + y^m$$

If both members of this equation are multiplied by $(x + y)$, the first member of the resulting equation will be $(x + y)^{m+1}$: to find what the second member will be, let us perform the multiplication, as indicated below:

Operation.

$$\begin{array}{l}
x^m + mx^{m-1}y + m\,.\,\frac{m-1}{2}x^{m-2}y^2 + \&c. + y^m \\
x + y \\ \hline
x^{m+1} + mx^m y + m\,.\,\frac{m-1}{2}x^{m-1}y^2 + \&c. + xy^m \\
\qquad\quad x^m y + \qquad\qquad mx^{m-1}y^2 + \&c. + mxy^m + y^{m+1} \\ \hline
\end{array}$$

$$x^{m+1} \begin{array}{c|} +m \\ +1 \end{array}\, x^m y \begin{array}{c|} +m\,.\,\frac{m-1}{2} \\ +m \end{array}\, x^{m-1}y^2 + \&c. \begin{array}{c|} +m \\ +1 \end{array}\, xy^m + y^{m+1}$$

But,

$$m \cdot \frac{m-1}{2} + m = m\left(\frac{m-1}{2} + 1\right) = \frac{(m+1)m}{1 \cdot 2};$$

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} + m \cdot \frac{m-1}{2} = m \cdot \frac{m-1}{2}\left(\frac{m-2}{3} + 1\right)$$
$$= \frac{(m+1)\,m(m-1)}{1 \cdot 2 \cdot 3};$$

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} + m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}$$
$$= \frac{(m+1)\,m(m-1)\,(m-2)}{1 \cdot 2 \cdot 3 \cdot 4}; \text{ and so on.}$$

Substituting these results in the product, we have,

$$(x+y)^{m+1} = x^{m+1} + (m+1)x^m y + \frac{(m+1)m}{1 \cdot 2}x^{m-1}y^2$$
$$+ \frac{(m+1)\,m(m-1)}{1 \cdot 2 \cdot 3} \cdot x^{m-2}y^3 + \text{\&c.}, + y^{m+1}.$$

If we examine the $(m+1)$ power, we see that the assumed laws of formation hold good in it. Hence, *if the assumed laws of formation hold good when the exponent of the power is m, they will also hold good when the exponent is $(m+1)$.*

Now, we have proved by actual multiplication, that the assumed laws hold good when the exponent is 5; hence, from what we have just proved, they will hold good when the exponent is 6. Hence, from the principle demonstrated, they must hold good when the exponent is 7; and if for 7, then for 8; if for 8, then for 9:

and so on, by successive deduction, it may be shown that the laws hold good for any whole number whatever.

If we denote any whole number by *n*, we shall have, from the preceding demonstration,

$$(x + y)^n = x^n + nx^{n-1}y + n\,.\,\frac{n-1}{2}x^{n-2}y^2$$
$$+ n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}x^{n-3}y^3 + \text{\&c.} + y^n,$$

which is the *binomial formula.*

We have only proved the truth of the formula when *n* is a positive whole number; it is, however, true when *n* is either positive or negative, entire or fractional, as will be demonstrated in the appendix.

If we change the places of *x* and *y*, we shall have, by the laws of formation,

$$(y + x)^n = y^n + ny^{n-1}x + n\,.\,\frac{n-1}{2}\,y^{n-2}x^2$$
$$+ n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}y^{n-3}x^3 + \text{\&c.} + x^n.$$

The second member of this equation is the same as the second member of the formula already deduced, taken in a reverse order. Comparing the two, we see that the coefficients taken in the same order are equal; this shows that *the coefficients of the second member of the binomial formula, at equal distances from the extremes, are equal.* Hence, in forming any power of a binomial, it is only necessary to find the coefficients to the middle of the development; the remaining ones can be written by taking these in a reverse order.

There is always one more term in the development than there are units in the exponent of the power; hence, an *odd* power of a binomial contains an *even* number of terms, and an *even* power contains an *odd* number of terms.

Method of Applying the Binomial Formula.

101. In applying the binomial formula to find any power of a binomial, we raise the first term to its successive powers as high as the n^{th}, and substitute them for the corresponding powers of x in the formula; we then raise the second term to its successive powers up to the n^{th}, and substitute them in the formula for the corresponding powers of y, substituting for n the exponent of the power.

EXAMPLES.

1. Find the third power of $a + b$.

Here, a takes the place of x, b the place of y, and 3 the place of n; making these substitutions in the formula, we have,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

2. Find the fourth power of $c + d$.

Here, c takes the place of x, d the place of y, and 4 the place of n; making these substitutions, we have,

$$(c + d)^4 = c^4 + 4c^3d + 6c^2d^2 + 4cd^3 + d^4.$$

3. Find the fifth power of $a + b$.

We may write the literal parts of each term of the development by the *law of exponents*, giving,

$$a^5,\ a^4b,\ a^3b^2,\ a^2b^3,\ ab^4, \text{ and } b^5.$$

The coefficients may be formed by the *law of coefficients*. The coefficient of the first term is 1, that of the second is 5, that of the third is $\frac{5 \times 4}{2}$, or 10, and the remaining coefficients are the same, taken in the reverse order; hence,

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

4. Find the sixth power of $(a + b)$.

Ans. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$.

5. Find the fifth power of $c + d$.

Ans. $c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5$.

6. Find the sixth power of $c + d$.

Ans. $c^6+6c^5d+15c^4d^2+20c^3d^3+15c^2d^4+6cd^5+d^6$.

7. Find the third power of $a - b$.

Here, a takes the place of x in the formula, $-b$ the place of y, and 3 the place of n, giving,

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

If the second power of the binomial is negative, all the odd terms are positive, and all the even terms negative.

8. Find the fourth power of $c - d$.

Ans. $c^4 - 4c^3d + 6c^2d^2 - 4cd^3 + d^4$.

9. Find the fifth power of $a - b$.

Ans $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

10. Find the sixth power of $c - d$.

Ans. $c^6-6c^5d+15c^4d^2-20c^3d^3+15c^2d^4-6cd^5+d^6$.

It is to be observed, that any power of the difference of two quantities may be written out by the two laws for exponents and coefficients, provided the signs of the terms be made alternately *plus* and *minus*, as in the above examples.

Cases in which the Terms have Numerical Coefficients.

102. If the terms of the given binomial have numerical coefficients, we may form any of its powers by means of the binomial formula:

11. Find the cube of $2a + 3b$.

Here, $2a$ takes the place of x in the formula, $3b$ the place of y, and 3 the place of n, giving,

$$(2a + 3b)^3 = (2a)^3 + 3(2a)^2(3b) + 3(2a)(3b)^2 + (3b)^3 \ldots (1);$$

or, performing the operations indicated,

$$(2a + 3b)^3 = 8a^3 + 36a^2b + 54ab^2 + 27b^3.$$

Simplification of the Operation.

103. If we examine the second member of equation (1), Art. 102, we see that each term is made up of three factors; 1st, a numerical factor; 2d, some power of $2a$; and 3d, some power of $3b$. The powers of $2a$ are arranged in descending order towards the right, the last term being the 0 power of $2a$, or 1; the powers of $3b$ are arranged in ascending order from the first term, which is the 0 power of $3b$, or 1.

The operation of raising a binomial, in which the terms have numerical coefficients, is most readily effected by writing the three factors of each term in a vertical column, and then performing the multiplications as indicated below:

$$\begin{array}{lllll}
\textit{Coefficients}; \quad . \; . \; . & 1 & +3 & +3 & +1 \\
\textit{Powers of } 2a; \quad . \; . & 8a^3 & +4a^2 & +2a & +1 \\
\textit{Powers of } 3b; \quad . \; . & 1 & +3b & +9b^2 & +27b^3 \\
\cline{2-5}
\qquad \therefore (2a+3b)^3 = & 8a^3 & +36a^2b & +54ab^2 & +27b^3
\end{array}$$

The preceding operation hardly requires explanation. In the first line, we write the numerical coefficients corresponding to the particular power taken from the formula; in the second line, we write the descending powers of the leading term down to the 0 power; in the third line, we write the ascending powers of the following term from the 0 power upwards. We then multiply each column from above downward.

It will be found most convenient to write the powers in the second line from right to left, beginning with the 0 power.

EXAMPLES.

12. Find the cube of $3a + 2b$.

OPERATION.

$$\begin{array}{lcccc}
\textit{Coefficients;} \;\ldots & 1 & +3 & +3 & +1 \\
\textit{Powers of } 3a; \;\ldots & 27a^3 & +9a^2 & +3a & +1 \\
\textit{Powers of } 2b; \;\ldots & 1 & +2b & +4b^2 & +8b^3 \\
\hline
\therefore (3a+2b)^3 = & 27a^3 & +54a^2b & +36ab^2 & +8b^3
\end{array}$$

13. Find the square of $7x - 3y$.

OPERATION.

$$\begin{array}{lccc}
\textit{Coefficients;} \;\ldots & 1 & +2 & +1 \\
\textit{Powers of } 7x; \;\ldots & 49x^2 & +7x & +1 \\
\textit{Powers of } -3y; \;\ldots & 1 & -3y & +9y^2 \\
\hline
\therefore (7x-3y)^2 = & 49x^2 & -42xy & +9y^2
\end{array}$$

14. Find the cube of $2x - 3y$.

OPERATION.

$$\begin{array}{lcccc}
\textit{Coefficients;} \;\ldots & 1 & +3 & +3 & +1 \\
\textit{Powers of } 2x; \;\ldots & 8x^3 & +4x^2 & +2x & +1 \\
\textit{Powers of } -3y; \;\ldots & 1 & -3y & +9y^2 & -27y^3 \\
\hline
\therefore (2x-3y)^3 = & 8x^3 & -36x^2y & +54xy^2 & -27y^3
\end{array}$$

15. Find the cube of $\frac{1}{2}x + \frac{1}{3}y$.

OPERATION.

$$\begin{array}{llllll}
\textit{Coefficients}; & \ldots & 1 & +3 & +3 & +1 \\
\textit{Powers of } \frac{1}{2}x; & \ldots & \frac{1}{8}x^3 & +\frac{1}{4}x^2 & +\frac{1}{2}x & +1 \\
\textit{Powers of } \frac{1}{3}y; & \ldots & 1 & +\frac{1}{3}y & +\frac{1}{9}y^2 & +\frac{1}{27}y^3 \\
\hline
\end{array}$$

$$\therefore \left(\frac{1}{2}x + \frac{1}{3}y\right)^3 = \frac{1}{8}x^3 + \frac{1}{4}x^2y + \frac{1}{6}xy^2 + \frac{1}{27}y^3$$

16. Find the fourth power of $\frac{1}{2}a - 3b$.

Ans. $\frac{1}{16}a^4 - \frac{3}{2}a^3b + \frac{27}{2}a^2b^2 - 54ab^3 + 81b^4$.

17. Find the cube of $2ax - 3by^2$.

Ans. $8a^3x^3 - 36a^2bx^2y^2 + 54ab^2xy^4 - 27b^3y^6$.

18. Find the fourth power of $\frac{ax}{b} - \frac{cy}{d}$.

Ans. $\frac{a^4x^4}{b^4} - \frac{4a^3cx^3y}{b^3d} + \frac{6a^2c^2x^2y^2}{b^2d^2} - \frac{4ac^3xy^3}{bd^3} + \frac{c^4y^4}{d^4}$.

19. Find the fourth power of $mx + ny$.

Ans. $m^4x^4 + 4m^3nx^3y + 6m^2n^2x^2y^2 + 4mn^3xy^3 + n^4y^4$.

20. Raise $a - 2x$ to the fourth power.

Ans. $a^4 - 8a^3x + 24a^2x^2 - 32ax^3 + 16x^4$.

Verify the following results:

21. $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$.

22. $\left(\frac{x}{a^2} - \frac{a^2}{x}\right)^3 = \frac{x^3}{a^6} - \frac{3x}{a^2} + \frac{3a^2}{x} - \frac{a^6}{x^3}$.

23. $(x^{pq} - 1)^3 = x^{3pq} - 3x^{2pq} + 3x^{pq} - 1$.

24. $(e^x - e^{-x})^3 = e^{3x} - e^{-3x} - 3(e^x - e^{-x})$.

25. $(5 - 4x)^4 = 625 - 2000x + 2400x^2 - 1280x^3 + 256x^4$.

26. $(5a^2c^2d - 4abd^2)^4 = 625a^8c^8d^4 - 2000a^7bc^6d^5$
$+ 2400a^6b^2c^4d^6 - 1280a^5b^3c^2d^7 + 256a^4b^4d^8$.

Powers of Polynomials.

104. The polynomial $a+b+c$, may be written under the form, $a+(b+c)$; hence, $(a+b+c)^n = [a+(b+c)]^n$; also, $(2a-x+3y+4z)^n$, equal to $[(2a-x)+(3y+4z)]^n$; and so on, for polynomials containing any number of terms.

To raise a polynomial to a power, we write it under the form of a binomial, each term of which may be a binomial, or some other polynomial; we then form the powers of these parts, and proceed in the same manner as with a true binomial:

EXAMPLES.

1. Find the cube of $a + (b + c)$.

OPERATION.

$$\begin{array}{lllllll}
1 + & 3 & + & 3 & + & 1 \\
a^3 + & a^2 & + & a & + & 1 \\
1 + & (b+c) & + & (b^2+2bc+c^2) & + & (b^3+3b^2c+3bc^2+c^3) \\
\hline
\multicolumn{6}{l}{a^3+(3a^2b+3a^2c)+(3ab^2+6abc+3ac^2)+(b^3+3b^2c+3bc^2+c^3)}
\end{array}$$

2. Find the square of $(2a - x) + (3y + 4z)$.

Ans. $4a^2-4ax+x^2+12ay+16az-6xy-8xz$
$+ 9y^2 + 24yz + 16z^2$.

Verify the following results:

3. $(1-2x+3x^2)^3 = 1 - 6x + 21x^2 - 44x^3 + 63x^4$
$-54x^5 + 27x^6$.

4. $(a^2 - a + \frac{1}{4})^2 = a^4 - 2a^3 + \frac{3}{2}a^2 - \frac{1}{2}a + \frac{1}{16}$.

5. $\left(x - \frac{1}{x} - 1\right)^3 = x^3 - \frac{1}{x^3} - 3x^2 - \frac{3}{x^2} + 5$.

6. $\left(7x^2 - \frac{x}{5} + 3\right)^2 = 49x^4 - \frac{14x^3}{5} + \frac{1051x^2}{25} - \frac{6x}{5} + 9$.

7. $\left(x + \frac{a}{3} - \frac{b}{2}\right)^2 = x^2 + \frac{2ax}{3} - bx - \frac{ab}{3} + \frac{b^2}{4} + \frac{a^2}{9}$.

8. $\left(x^2 - x + \frac{1}{4}\right)^2 = x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}$.

9. $(3a^2 - 2ab + 5b^2)^2 = 9a^4 - 12a^3b + 34a^2b^2$
$- 20ab^3 + 25b^4$.

10. $(2x^2 - 3x + 4)^2 = 4x^4 - 12x^3 + 25x^2 - 24x + 16$.

11. $(x^3 + 2x^2 + 3x + 4)^2 = x^6 + 4x^5 + 10x^4 + 20x^3$
$+ 25x^2 + 24x + 16$.

CHAPTER VII.

EXTRACTION OF ROOTS.

Definitions and Symbols.

105. A **root** of a quantity is one of its equal factors. If a quantity is resolved into 2 equal factors, one of them is called the *square root*: the symbol for the square root is $\sqrt{\ }$.

If a quantity is resolved into 3 equal factors, one of these factors is called the *cube root*: the symbol for the cube root is $\sqrt[3]{\ }$.

If a quantity is resolved into n equal factors, one of these factors is called the n^{th} *root*: the symbol for the n^{th} root is $\sqrt[n]{\ }$.

The operation of finding one of the n equal factors of a quantity (n being a positive whole number), is called **extracting its n^{th} root.**

Another Method of Indicating Roots.

106. Instead of employing the symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[n]{\ }$, to indicate the *square, cube, n^{th} roots,* it is more convenient to employ the fractional exponents, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{n}$, which indicate the same thing. Thus,

$$\sqrt{a} = a^{\frac{1}{2}}, \quad \sqrt[3]{a} = a^{\frac{1}{3}}, \quad \sqrt[4]{a} = a^{\frac{1}{4}}, \quad . \ . \ . \quad \sqrt[n]{a} = a^{\frac{1}{n}}.$$

It will be shown hereafter, that quantities having fractional exponents may be operated on by the same rules as when they have entire exponents.

Square Root of a Number.

107. The **square root** of a number is one of its two equal factors. Thus, $25 = 5 \times 5$; hence, 5 is the square root of 25; that is, $\sqrt{25} = 5$, or $(25)^{\frac{1}{2}} = 5$.

The following table, verified by actual multiplication, is employed in finding the square root of any number less than 100.

TABLE.

1	4	9	16	25	36	49	64	81	100.	*Powers.*
1	2	3	4	5	6	7	8	9	10.	*Roots.*

To employ the table for finding the square root of a number less than 100. *Look for the number in the first line; if it is found there, its square root will be found immediately under it; if it is not found there, it will fall between two numbers in that line, and its square root will be found between the two numbers immediately below; the less number of the two will be the entire part of the root, and will be the true root to within less than 1.*

If a number is greater than 100, its square root will be greater than 10, that is, it will contain *tens* and *units*. Let N denote such a number, x the tens of its square root, and y the units; then will

$$N = (x + y)^2 = x^2 + 2xy + y^2 = x^2 + (2x + y)y;$$

that is, the number is equal to the *square of the tens* in its root, plus *twice the product of the tens by the units,* plus *the square of the units.*

We first find the *tens* of the root. Since the square of tens can contain no significant figure less than *hundreds,* the two figures on the right may be pointed off, and the square of the tens will be found in the number to the left of the point. If we subtract the square of the tens from the given number, the remainder will be equal to twice the product of the tens by the units, plus the square of the units. Consequently, if we divide the remainder by twice the tens, the quotient will give the units, or a number greater than the units. To test it, add it to twice the tens, and multiply the sum by the quotient found; if the product is equal to, or less than the remainder, the number found is the root sought; but if greater, diminish the last figure by 1, and test as before, till the correct number is found.

EXAMPLE.

Find the square root of 1764.

Pointing off the two right hand figures, there remains the number 17, the greatest perfect square in which is 16. The square root of 16, or 4, is therefore the number of *tens* of the required root.

OPERATION.

$$\begin{array}{r|l|l}
 & 17{}^{\cdot}64 & \underline{42} \\
 & 16 & \\
\underline{82} & \overline{16\,4} & \\
 & \underline{164} & \\
 & 0 &
\end{array}$$

Place the figure on the right after the manner of a quotient. Subtracting 16 *hundreds* (the square of 4 *tens*) from the given number, we have 164 for a remainder. Doubling the tens, we have 8 *tens*, which is contained in 164 (that is, 8 is contained in 16) 2 *times;* adding 2 to 8 *tens*, that is, annexing 2 to 8 and multiplying the result by 2, we find 164, which is equal to the remainder already found; hence, the required root is 42.

If the given number contains more than four places, we point off a period of 2 figures from the right, for the same reason as before. The operation is then reduced to finding the square root of the remaining numbers, that is, the *tens* of the root. In finding this root, for the same reason as before, we point off another period of two figures, and the operation then is reduced to finding the square root of the remaining number, that is, the *tens* of the *tens*, or the *hundreds* of the root. If the number on the left of the second point, contains more than two figures, we again point off a period of two figures, and so on continually: the operation is then reduced to a successive repetition of that already explained; hence, we have the following rule for extracting the square root of a number:

RULE.

I. Point the number off into periods of two figures each, beginning at the units' place.

II. Find the greatest perfect square in the first period on the left, and place its square root on the right, after the manner of a quotient in a division; then subtract the square of this number *from the first period, and bring down the next period for a remainder.*

III. Double the root already found, and see how often it is contained in this remainder, exclusive of the right hand figure; write this quotient for a second figure of the root, annex it also to the divisor used; multiply the divisor thus increased by the quotient already found, subtract this product from the first remainder, and bring down the next period for a second remainder.

IV. Double the root already found and proceed as before, continuing the operation till every period has been employed. If the final remainder is 0, the root is exact, if it is not 0, the root found is true to within less than 1.

EXAMPLES.

1. Find the square root of 273529.

OPERATION.

$$\begin{array}{rl}
 & 2\dot{7}\ 3\dot{5}\ 29 \;\big|\; \underline{523} \\
 & \underline{25} \\
\underline{10\,2} & \big|\ 23\,5 \\
 & \underline{20\,4} \\
\underline{104\,3} & \big|\ 312\,9 \\
 & \underline{312\,9} \\
 & \quad 0.
\end{array}$$

2. Find the square root of 61009. *Ans.* 247.

Find the square roots of the following numbers:

3. 4096. *Ans.* 64.

4. 582169. *Ans.* 763.

5. 956484. *Ans.* 978.

6. 57198969. *Ans.* 7563.

7. 68492176. *Ans.* 8276.

8. 1018081. *Ans.* 1009.

9. 9803161. *Ans.* 3131.

10. 1522756. *Ans.* 1234.

Square Root of a Common Fraction.

108. A fraction may be squared by squaring its numerator and denominator separately (Art. 96): reversing the principle, we have the following rule for extracting the square root of a fraction:

RULE.

Extract the square root of the numerator for a new numerator, and the square root of the denominator for a new denominator.

EXAMPLES.

Extract the square root of the following fractions:

1. $\frac{25}{36}$. *Ans.* $\frac{5}{6}$.

2. $\frac{256}{625}$. *Ans.* $\frac{16}{25}$.

3. $\frac{98}{242}$, or $\frac{49}{121}$. *Ans.* $\frac{7}{11}$.

4. $\frac{392}{18}$, or $\frac{196}{9}$. *Ans.* $\frac{14}{3}$.

5. $\frac{2209}{196}$. *Ans.* $\frac{47}{14}$.

6. $54\frac{19}{25}$, or $\frac{1369}{25}$. *Ans.* $7\frac{2}{5}$.

Square Roots by Approximation.

109. If the terms of the fraction, after being reduced to its simplest form, are not perfect squares, the exact root is impossible. In such cases, we may multiply both terms by any number that will make the denominator a perfect square. Then, extracting the square root of the numerator to the nearest unit for a new numerator, and the square root of the denominator for a new denominator, the resulting fraction will be the true root to within less than the fractional unit of the root.

EXAMPLES.

Find the square roots of the following fractions, approximately:

1. $\frac{19}{8} = \frac{38}{16}$. *Ans.* $\frac{6}{4}$ to within $\frac{1}{4}$.

2. $\frac{19}{8} = \frac{19 \times 32}{8 \times 32}$. *Ans.* $\frac{24}{16}$ to within $\frac{1}{16}$.

By increasing the factor introduced in both terms, we may make the resulting root true to any degree of exactness.

3. $\frac{38}{5} = \frac{190}{25}$. *Ans.* $\frac{14}{5}$ to within $\frac{1}{5}$.

4. $\frac{45}{7} = \frac{45 \times 343}{7 \times 343}$. *Ans.* $\frac{124}{49}$ to within $\frac{1}{49}$.

5. $\frac{57}{8} = \frac{57 \times 20000}{8 \times 20000}$. *Ans.* $\frac{1067}{400}$ to within $\frac{1}{400}$.

6. $\frac{21}{2} = \frac{4200}{400}$. *Ans.* $\frac{65}{20}$ to within $\frac{1}{20}$.

Square Root of a Decimal Fraction.

110. The denominator of a decimal fraction is a perfect square when the number of decimal places is even, and the number of decimal places in the root of the decimal is half the number of decimal places in the given decimal; hence, from the preceding principle, we have the following rule for finding the square root of any decimal fraction, to any degree of exactness:

RULE.

Annex 0's to the decimal till the number of decimal places is twice the number of decimal places required in the root; extract the square root of the result as though it were a whole number, and point off the required number of decimal places in the root.

A vulgar fraction may be converted into a decimal, and then the above rule may be applied. The rule is also applicable to the case of whole numbers, to which we may annex any number of decimal 0's.

EXAMPLES.

Find the square roots of the following decimals, approximately:

1. 9.6 = 9.600000. *Ans.* 3.098 to within .001.
2. 7.65 = 7.65000000. *Ans.* 2.7658 to within .0001.
3. 15.2379. *Ans.* 3.90357 to within .00001.
4. $\frac{4}{7}$, or .571428. *Ans.* 0.755 to within .001.
5. $\frac{1}{17}$. *Ans.* 0.24253 to within .00001.
6. $10\frac{3}{10}$. *Ans.* 3.209 to within .001.
7. 5, or 5.000000. *Ans.* 2.236 to within .001.

8. 22. *Ans.* 4.69 to within .01.

9. 153. *Ans.* 12.36931 to within .00001.

10. 101. *Ans.* 10.04987 to within .00001.

Cube Root of Numbers.

111. The **cube root** of a number is one of its three equal factors. Thus, $27 = 3 \times 3 \times 3$; hence, 3 is the cube root of 27; also, 4 is the cube root of 64, because $4 \times 4 \times 4 = 64$.

The following table, verified by actual multiplication, is employed in finding the cube root of any number less than 1000:

TABLE.

1	8	27	64	125	216	343	512	729	1000.
1	2	3	4	5	6	7	8	9	10.

To employ the table in finding the cube root of a number less than 1000. *Look for the number in the first line, if found there, its cube root is immediately below it; if the number is not in the first line, it will fall between two numbers in that line, and its root will fall between the corresponding numbers in the second line; the less number of the two will be the entire part of the required cube root, that is, it is the cube root to within less than 1.*

The cube of 10 is 1,000; the cube of 2 *tens*, or 20, is 8,000; the cube of 3 *tens*, or 30, is 27,000; and so on up to the cube of 10 *tens*, or 100, which is 1,000,000.

If a number is greater than 1000 its cube root is greater than 10, that is, it is made up of *tens* and *units*. Let N denote a number greater than 1000, and let the *tens* of its cube root be denoted by a and the *units* of its cube root by b; we shall have,

$$N = (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

which can be placed under the form

$$N = a^3 + [3a^2 + (3a + b)\,b] \times b. \qquad (1)$$

This formula indicates a method of finding the cube root of a number greater than 1000. As an illustration of this method let it be required to find the cube root of 405,224.

OPERATION.

$$\begin{array}{lr|l}
 & & 405\;\dot{2}24 \;|\, \underline{74} \\
 & & 343 \\
\cline{3-3}
\textit{Trial Divisor.}\quad 3 \times 70^2 = & 14700 & \;\;62\,224 \\
4 \times (3 \times 70 + 4) = & \underline{856} & \\
\textit{Complete Divisor.}\;\;\ldots & 15556 & \;\;\underline{62\,224} \\
\cline{2-3}
 & & \qquad 0
\end{array}$$

EXPLANATION.—Because the *cube of tens* contains no significant figure of a less denomination than *thousands*, we point off a period of three figures from the right; the cube of the *number of tens* in the root will be contained in the remaining period, that is, in 405 We see from the table that the greatest perfect cube contained in 405 is 343, whose cube root is 7; we therefore write 7 for the number of tens of the required root. We next subtract 343 from 405 and to the remainder we annex the second period, giving 62224, which we regard as a dividend. We see from formula (1) that this dividend is composed of two factors, one of which is the *units of the root* and the other is *a number greater than 3 times the square of the tens of the root;* hence, if we divide it by *3 times the square of the tens*, the quotient will be *the units* of the root or some greater number; this quotient, which we call a *trial figure*, must be tested in the manner indicated by the formula. In the present example, 3 times the square of the tens is 3×70^2, or 14700, and the trial figure is 4. To test this figure we form the complete divisor; this we do by adding to the trial divisor 4 times the result obtained by increasing 3 times the tens of the root by the trial figure. This gives the *complete divisor* 15556, and this multiplied by 4 gives 62224, which taken from the dividend gives 0 for a remainder. Hence, the required root is 74.

If the product of the completed divisor in any case is greater than the dividend, we diminish the trial figure by 1 and test as before, and so on till the product is equal to or less than the dividend. In the former case the root is *exact*, in the latter case it is true to within less than 1.

If the number to the left of the first period on the right contains more than three figures, the tens of the root will be made up of *tens of tens*, or *hundreds*, and *units of tens*, or *tens*; and for the same reason as before, a second period of these figures must be pointed off from the right, and so on until the period on the left contains three figures or less. The root is then found by a continued repetition of the process above given; hence the following

RULE.

I. Separate the given number into periods of three figures each, counting from the units' place; the period on the left may contain less than three figures.

II. Find from the table the greatest perfect cube in the first period on the left, and write its root for the first figure of the required root; subtract this cube from the first period, and to the remainder annex the following period for a dividend.

III. Multiply the square of the root already found by 3, and to the product annex two ciphers for a trial divisor; find how many times this is contained in the dividend, and write the quotient for a trial figure of the root; then annex this trial figure to 3 times the root previously found, multiply this result by the trial figure, and add the resulting product to the trial divisor for a complete divisor.

IV. Multiply the divisor thus completed by the trial figure of the root, subtract the product from the dividend, and to the remainder annex the following period for a new dividend.

V. Proceed as before, continuing the operation till all the periods have been used.

NOTES.—1. If a trial figure proves too great, diminish it successively by 1's till it is correct.

2. If the last remainder is 0, the number is a perfect cube, and the root is exact; if not, the root is true to within less than 1.

EXAMPLES.

1. Find the cube root of 111,980,168.

OPERATION.

$$\begin{array}{lr|l}
 & & 111\dot{}980\dot{}168 \;|\; \underline{482} \\
 & & 64 \\ \hline
\textit{1st Trial Divisor} \;.\;.\;. & 3 \times 40^2 = 4800 & 47\,980 \\
 & 8 \times (3 \times 40 + 8) = \underline{1024} & \\
\textit{1st Complete Divisor} \;.\;.\;.\;.\;.\;. & 5824 & 46\,592 \\ \hline
\textit{2d Trial Divisor} \;.\;. & 3 \times 480^2 = 691200 & 1\,388\,168 \\
 & 2 \times (3 \times 480 + 2) = \underline{2884} & \\
\textit{2d Complete Divisor} \;.\;.\;.\;.\;.\;. & 694084 & \underline{1\,388\,168} \\
 & & 0
\end{array}$$

NOTE.—Here we have three periods. In the first place we find the cube root of the first two periods to within 1, as explained before;

in doing this we find for the first trial figure the number 9, which on being tested proves too great; we then try 8 and find that it is correct; we then proceed as before, using 48 as the part of the root already found.

2. Find the cube root of 224755712.	*Ans.* 608.
3. Find the cube root of 2460375.	*Ans.* 135.
4. Find the cube root of 11089567.	*Ans.* 223.
5. Find the cube root of 40353607.	*Ans.* 343.
6. Find the cube root of 403583419.	*Ans.* 739.
7. Find the cube root of 115501303.	*Ans.* 487.

Simplification.

112. If the index of the required root is composed of two factors, the operation of extracting the root may be simplified.

Let N be any number, and assume

$$\sqrt[m]{\sqrt[n]{N}} = r \quad . \quad . \quad . \quad . \quad (1)$$

Raising both members of (1) to the m^{th} power, we have,

$$\sqrt[n]{N} = r^m \quad . \quad . \quad . \quad . \quad (2)$$

Raising both members of (2) to the n^{th} power, we have,

$$N = r^{mn} \quad . \quad . \quad . \quad . \quad (3)$$

Extracting the mn^{th} root of both members of (3), we have,

$$\sqrt[mn]{N} = r \quad . \quad . \quad . \quad . \quad (4)$$

Things which are equal to the same thing are equal to each other; hence, placing the first members of (1) and (4) equal, we have,

$$\sqrt[mn]{N} = \sqrt[m]{\sqrt[n]{N}}.$$

From which, we conclude that *the* mn^{th} *root of any number is equal to the* m^{th} *root of the* n^{th} *root of that number.*

We may, therefore, factor the index and extract the root of the number that is indicated by one of the factors, and then the root of the result that is indicated by the other factor. It will be simpler to begin with the least factor.

EXAMPLES.

1. Find the fourth root of 923521.

We have, $\sqrt{923521} = 961,$

and, $\sqrt{961} = 31.$

Hence, 31 is the required root.

2. Find the sixth root of 191102976.

We have, $\sqrt{191102976} = 13824,$

and $\sqrt[3]{13824} = 24.$ *Ans.*

3. Find the fourth root of 65536. *Ans.* 16.

Higher Roots of Fractions.

113. It has been shown (Art. 96), that a fraction may be raised to any power by raising the numerator to that power for a new numerator, and the denominator to that power for a new denominator; reversing this principle, we have the following rule for finding any root of a fraction;

RULE.

Extract the required root of the numerator for a new numerator, and the same root of the denominator for a new denominator.

EXAMPLES.

1. Find the cube root of $\frac{8}{27}$. *Ans.* $\frac{2}{3}$.
2. Find the cube root of $\frac{64}{125}$. *Ans.* $\frac{4}{5}$.
3. Find the cube root of $\frac{125}{13824}$. *Ans.* $\frac{5}{24}$.
4. Find the fourth root of $29\frac{52}{81}$. *Ans.* $2\frac{1}{3}$.
5. Find the fourth root of $104\frac{536}{625}$. *Ans.* $3\frac{1}{5}$.
6. Find the sixth root of $11\frac{25}{64}$. *Ans.* $1\frac{1}{2}$.

Higher Roots by Approximation.

114. If the denominator of a fraction is not an exact power of the degree indicated, we may multiply both terms of the fraction by such a number as will make the denominator an exact power of that degree. Then, extracting the required root of the resulting numerator to within less than 1, and writing the result over the required root of the denominator of the fraction, the result will be the true root, to within less than the fractional unit.

EXAMPLES.

Find the cube roots of the following numbers approximately:

1. $\frac{173}{32} = \frac{346}{64}$. *Ans.* $\frac{7}{4}$ to within $\frac{1}{4}$

2. $\frac{125}{256} = \frac{250}{512}$. *Ans.* $\frac{6}{8}$ to within $\frac{1}{8}$

3. $278 = \frac{278000}{1000}$. *Ans.* $\frac{65}{10}$, or 6.5 to within .1.

4. Find the fourth root of $\frac{210}{2187} = \frac{630}{6561}$.

Ans. $\frac{5}{9}$ to within $\frac{1}{9}$.

Higher Roots of Decimals.

115. The denominator of a decimal fraction is a perfect n^{th} power (n being any whole number), when the number of its decimal places is divisible by n, and the number of decimal places in its n^{th} root is the n^{th} part of the number of decimal places in the given decimal; hence, from preceding principles, we have the following rule for finding the n^{th} root of a decimal fraction to any desired degree of accuracy:

RULE.

Annex 0's to the given decimal, till the number of decimal places is n times the number in the required root; extract the n^{th} root of the result, as though it were a whole number, and point off the required number of decimal places in the root.

The rule is applicable to a vulgar fraction, for we may convert it into a decimal by known rules; it it is also applicable to finding an approximate root of a whole number.

EXAMPLES.

Find the cube roots of the following numbers, approximately:

1. 5.8. *Ans.* 1.7967 to within .0001.
2. 102.875. *Ans.* 4.6856 to within .0001.
3. $\frac{2}{3}$. *Ans.* 0.873 to within .001.
4. $\frac{5}{6}$. *Ans.* 0.941 to within .001.
5. 82. *Ans.* 4.344 to within .001.
6. 550. *Ans.* 8.193 to within .001.
7. Find the fourth root of 72.
   Ans. 2.91 to within .01.
8. Find the sixth root of 28.25.
   Ans. 1.745 to within .001.

Find the fourth roots of the following numbers:

9. $\frac{7}{4}$. *Ans.* 1.15 to within .01.
10. $8\frac{15}{49}$. *Ans.* 1.69 to within .01.
11. 13. *Ans.* 1.89 to within .01.
12. Find the cube root of 58230.6.
    Ans. 38.76 to within .01.

Roots of Monomials.

116. We have seen, (Art. 96), that a monomial can be raised to any power by raising the coefficient to the required power for a new coefficient, and giving to each letter an exponent equal to its original exponent, multiplied by the exponent of the required power; reversing this principle, we have the following rule for extracting any root of a monomial:

RULE.

Extract the required root of the coefficient for a new coefficient; after this, write all the letters, giving to each an exponent equal to its original exponent, divided by the index of the required root.

This rule, combined with that for extracting any root of a fraction, enables us to extract any root of a monomial, whether entire or fractional.

EXAMPLES.

Find the square roots of the following monomials:

1. $9a^2b^4x^2$. *Ans.* $3ab^2x$.

2. $49a^2x^4y^6$. *Ans.* $7ax^2y^3$.

3. $\frac{a^4y^2z^8}{16}$. *Ans.* $\frac{a^2yz^4}{4}$.

4. $\frac{25a^4x^4y^2}{81a^2b^2c^4}$. *Ans.* $\frac{5a^2x^2y}{9abc^2}$.

5. $25a^{-2}b^{-4}c^2$ *Ans.* $5a^{-1}b^{-2}c$.

Find the cube roots of the following monomials:

6. $8a^3b^6y^6$. *Ans.* $2ab^2y^2$.

7. $\dfrac{8a^3y^3}{27d^6z^6}$. *Ans.* $\dfrac{2ay}{3d^2z^2}$.

8. Find the fourth root of $\dfrac{16a^4y^4}{625z^8}$. *Ans.* $\dfrac{2ay}{5z^2}$.

9. Find the cube root of $343x^{-3}y^{-6}$. *Ans.* $7x^{-1}y^{-2}$.

Rule for Signs of Roots.

117. Since the square of $+a$ is a^2, and the square of $-a$ is also a^2, it follows that a^2 has two square roots, $+a$ and $-a$. Further, since every even power of a positive quantity is equal to the same power of that quantity taken with a negative sign, it follows that every positive quantity has two square roots, two fourth roots, two sixth roots, &c., which are equal numerically, but have contrary signs. Thus,

$$\sqrt{25a^2b^2} = \pm 5ab;\ \sqrt[4]{16a^4b^8} = \pm 2ab^2;\ \sqrt[6]{a^{12}b^6} = \pm a^2b.$$

Since every odd power of a quantity has the same sign as the quantity, it follows that the sign of any odd root is the same as the sign of the quantity; hence, the following rule for signs:

RULE.

Every even root of a positive quantity must have the double sign, ±; every odd root of any quantity must have the sign of the quantity.

In the preceding examples, only the numerical value of the roots have been considered; hereafter, the proper signs will be prefixed to the results.

EXAMPLES.

1. Find the cube root of $-27a^3x^{-3}y^6$.

   Ans. $-3ax^{-1}y^2$.

2. Find the square root of $\frac{64}{9}a^4b^4$. *Ans.* $\pm\frac{8}{3}a^2b^2$.

3. Find the 4th root of $a^{4m}x^{8m}y^{4m}$. *Ans.* $\pm a^mx^{2m}y^m$.

4. Find the cube root of $-512a^{-3}$. *Ans.* $-8a^{-1}$.

5. Find the square root of $\frac{256a^2b^4}{9x^4y^2}$.

   Ans. $\pm\frac{16ab^2}{3x^2y}$.

6. Find the cube root of $\frac{729x^3y^{-3}}{a^6b^{-6}}$.

   Ans. $+\frac{9xy^{-1}}{a^2b^{-2}}$.

7. Find the square root of $900a^2x^4y^6$.

   Ans. $\pm 30ax^2y^3$.

8. Find the 4th root of $\frac{81a^4x^4}{y^8}$. *Ans.* $\pm\frac{3ax}{y^2}$.

Definition of an Imaginary Quantity.

118. It is impossible that an even power of a quantity, either positive or negative, should be a negative quantity; hence, it is equally impossible to extract any even root of a negative quantity. An *indicated* even root of a negative quantity, is called an **imaginary quantity**. Thus, $\sqrt{-4}$, $\sqrt{-a^2}$, $\sqrt[4]{-b^2}$, are imaginary quantities.

Square Root of Polynomials.

119. To deduce a rule for extracting the square root of a polynomial, let us suppose the root to be known, and to be arranged with respect to some letter. We may regard the root as composed of *the first term* plus *the sum of all the other terms.* Hence, its square, which is the given polynomial, will be made up of *the square of the first term,* plus *twice the product of the first term by the sum of all the other terms,* plus *the square of the sum of all the other terms,* (Art. 106). Now the square of the first term of the root must be of a higher degree with respect to the leading letter than any other term. Hence, if we arrange the given polynomial with respect to any letter, the square root of the first term will be the first term of the required square root. If the square of this term of the root is subtracted from the given polynomial, and the first term of the remainder divided by twice the first term of the root, the quotient will be the second term of the root. The rest of the operation for finding the square root, is entirely analogous to that for finding the square root of a whole number, and will be best understood from an example. Let it be required to extract the square root of $9x^4 + 12x^3 + 28x^2 + 16x + 16$:

OPERATION.

$$
\begin{array}{r|l|l}
 & 9x^4 + 12x^3 + 28x^2 + 16x + 16 & \underline{3x^2 + 2x + 4} \\
 & \underline{9x^4 } & \\
6x^2 + 2x & 12x^3 + 28x^2 + 16x + 16 & \ldots \text{ 1st } \textit{remainder.} \\
 & \underline{12x^3 + 4x^2 } & \\
\underline{6x^2 + 4x + 4} & 24x^2 + 16x + 16 & \ldots \text{ 2d } \textit{remainder.} \\
 & \underline{24x^2 + 16x + 16} & \\
 & \qquad 0 & \ldots \text{ 3d } \textit{remainder.}
\end{array}
$$

The polynomial having been arranged with respect to x, the square root of the first term is $3x^2$. Subtracting $9x^4$ from the given polynomial, and dividing the first term of the remainder by $6x^2$, which is double $3x^2$, we find $2x$ for the second term, which we add to the root and also to the divisor. Multiplying the divisor, thus augmented, by $2x$, and subtracting the product from the first remainder, we have a second remainder. Doubling the root already found, and dividing the first term of the second remainder by the first term of the last divisor, we find 4 for the third term. Adding this to the root and also to the second divisor, and multiplying the divisor by the last term found, we find for the final remainder 0; hence, the required root is $3x^2 + 2x + 4$.

In the same way, the square root of any polynomial may be found; hence, the following

RULE.

I. Arrange the polynomial with reference to some letter, and extract the square root of the first term for the first term of the root. Subtract the square of this term from the polynomial for the first remainder.

II. Double the root already found, and place it on the left of the first remainder for a divisor; divide the first term of the first remainder by this divisor, for the second term of the root; add the quotient to the root found, and also to the divisor; multiply the divisor, thus augmented, by the last term of the root found, and subtract *the product from the first remainder for a second remainder.*

III. Double the root already found for a second divisor. Divide the first term of the second remainder by the first term of the second divisor for the third term of the root; add this term to the root and to the second divisor, and proceed as before, continuing the operation as far as desirable.

If a remainder is found, equal to 0, the root is exact.

EXAMPLES.

1. Extract the square root of $a^4 - 2a^3 + 3a^2 - 2a + 1$.

OPERATION.

$$\begin{array}{r|l}
 & a^4 - 2a^3 + 3a^2 - 2a + 1 \;\big|\; \underline{a^2 - a + 1} \\
 & \underline{a^4 \qquad\qquad\qquad} \\
\underline{2a^2 - a} & -2a^3 + 3a^2 \\
 & \underline{-2a^3 + a^2 \qquad} \\
\underline{2a^2 - 2a + 1} & 2a^2 - 2a + 1 \\
 & \underline{2a^2 - 2a + 1} \\
 & \qquad 0
\end{array}$$

In finding the several remainders, all of the terms need not be brought down; only as many as are needed.

Extract the square roots of the following polynomials:

2. $9x^2 - 30ax + 25a^2 + 5a^3 + \frac{a^4}{4} - 3a^2x$.

Ans. $3x - 5a - \frac{a^2}{2}$.

3. $4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4.$

Ans. $2x^2 + 2ax + 4b^2.$

4. $\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2\left(\frac{a}{b} + \frac{b}{a}\right) + 3.$ *Ans.* $\frac{a}{b} + \frac{b}{a} + 1.$

5. $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} + \frac{xy}{3} - \frac{xz}{4} - \frac{yz}{6}.$ *Ans.* $\frac{x}{2} + \frac{y}{3} - \frac{z}{4}.$

6. $\frac{x^2}{9} + \frac{4y^2}{25} + \frac{z^2}{16} + \frac{4xy}{15} - \frac{xz}{6} - \frac{yz}{5}.$

Ans. $\frac{x}{3} + \frac{2y}{5} - \frac{z}{4}.$

7. $x^4 + 2px^3 + (p^2 - 2q)x^2 - 2pqx + q^2.$

Ans. $x^2 + px - q.$

8. $(x + x^{-1})^2 - 4(x - x^{-1}).$ *Ans.* $x - x^{-1} - 2.$

9. $9a^{2m} + 6a^{3m+1} + 25c^{2m-4} - 30a^mc^{m-2} + a^{4m+2}$
$-10a^{2m+1}c^{m-2}.$ *Ans.* $3a^m - 5c^{m-2} + a^{2m+1}.$

10. $49x^4 + 9 - \frac{14x^3}{5} - \frac{6x}{5} + \frac{1051x^2}{25}.$

Ans. $7x^2 - \frac{x}{5} + 3.$

11. $1 + x.$ *Ans.* $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \text{\&c.}$

12. $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2} + 1\right) + \frac{4y^2}{x^2}\left(\frac{y^2}{x^2} + 1\right) + 3.$

Ans. $\frac{x^2}{2y^2} + \frac{2y^2}{x^2} + 1.$

Higher Roots of Polynomials.

120. We may find the cube root of a polynomial by a method entirely analogous to that employed in finding the cube root of a number, but there is a simpler method of proceeding. The dividend found at each step of the operation is the result obtained by subtracting the *cube of the root already obtained* from *the given quantity*. But we only need the first term of this dividend and the first term of the divisor to find the next term of the root. The method of proceeding is shown in the following example.

Let it be required to find the cube root of

$$x^6 + 6x^5 - 40x^3 + 96x - 64.$$

OPERATION.

$$\begin{array}{rl|l} & x^6+6x^5-40x^3+96x-64 & x^2+2x-4 \\ \cline{3-3} (x^2+2x)^3 = & x^6+6x^5+12x^4+8x^3 & 3x^4 \ . \ \textit{Divisor.} \\ \cline{2-3} \textit{2d Dividend.} \ \ . \ \ . \ \ . & -12x^4-48x^3, \text{ etc.} & \\ \cline{2-2} (x^2+2x-4)^3 = & x^6+6x^5-40x^3+96x-64 & \\ \cline{2-2} & \qquad\qquad 0 & \end{array}$$

EXPLANATION.—The cube root of the first term, x^6, is x^2; this is the first term of the required root. Subtracting the cube of x^2 from the given polynomial, we have for the first term of the remainder $6x^5$, which need not be brought down. Multiplying the square of the first term of the root by 3, we have for a divisor $3x^4$ (which will be the first term of all subsequent divisors). Dividing $6x^5$ by $3x^4$ we find $2x$ for the second term of the root. Subtracting the cube of $x^2 + 2x$ from the given polynomial, we find $-12x^4$ for the first term of the second dividend. Dividing this by $3x^4$ we find -4 for the third term of the root. Subtracting the cube of $x^2 + 2x - 4$ from the given polynomial, we find 0 for a remainder; hence the required root is $x^2 + 2x - 4$.

Other cases may be treated in the same manner; hence, the

RULE.

I. Arrange the given polynomial with reference to one of its letters, and extract the cube root of the first term; this will be the first term of the root.

II. Divide the second term of the polynomial by

3 times the square of the first term of the root; the quotient will be the second term of the root.

III. Subtract the cube of the first two terms of the root from the given polynomial, and divide the first term of the remainder by 3 times the square of the first term of the root; the quotient will be the third term of the root.

IV. Continue this operation till a remainder is found equal to 0, or until it is shown that the polynomial is not a perfect cube.

EXAMPLES.

Find the cube roots of the following polynomials:

1. $8x^3 - 12x^2 + 6x - 1$. *Ans.* $2x - 1$.
2. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$. *Ans.* $x^2 - 2x + 1$.
3. $64a^6 - 288a^5 + 1080a^3 - 1458a - 729$. *Ans.* $4a^2 - 6a - 9$.
4. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$. *Ans.* $1 - 2x + 3x^2$.

By extracting the required root of the first and last terms, two terms of the root may be found, from which the remaining ones may, sometimes, be determined by inspection; the whole root may then be verified as above.

To find the fourth root of a polynomial extract its square root and then extract the square root of the result.

Let it be required to find the fourth root of

$$16x^4 - 128x^3y + 384x^2y^2 - 512xy^3 + 256y^4.$$

Extracting its square root, we have $4x^2 - 16xy + 16y^2$; and the square root of this polynomial gives $2x - 4y$; which is the required root.

CHAPTER VIII.

RADICALS.

I. TRANSFORMATION OF RADICALS.

Definitions.

121. A **radical** is an indicated root of an imperfect power of the degree indicated. Thus, $\sqrt{3}$ and $\sqrt[3]{4}$ are radicals; they are also called **irrational quantities** or **surds.**

An indicated root of a perfect power of the degree indicated, as $\sqrt{9}$, is a rational quantity under a radical form.

The **coefficient** of a radical is the factor without the radical sign. Thus, in the expression, $3\sqrt{3}$, 3 is the coefficient; in the expression $\sqrt{3}$, 1 is the coefficient.

122. Radicals are of different **degrees**, the degree being determined by the index of the radical. Thus, $\sqrt{3}$ is a radical of the *second* degree; $\sqrt[3]{4}$ is a radical of the *third* degree, and $\sqrt[n]{a}$ is a radical of the n^{th} degree.

123. Radicals are **similar** when the radical parts are alike, that is, when they are of the same degree and when the quantities under the radical sign are the same. Thus, $3\sqrt[3]{ab}$ and $5\sqrt[3]{ab}$ are similar.

Notation.

124. It has already been explained that radical quantities can be written by means of fractional exponents. The following table indicates the conventional methods of expressing radicals, powers, and reciprocals:

TABLE OF EQUIVALENT EXPRESSIONS.

$\sqrt[3]{x}$,	$\sqrt[n]{y}$,	equivalent to	$x^{\frac{1}{3}}$,	$y^{\frac{1}{n}}$.
$\sqrt[3]{x^2}$,	$\sqrt[n]{y^m}$,	equivalent to	$x^{\frac{2}{3}}$,	$y^{\frac{m}{n}}$.
x^{-3},	y^{-n},	equivalent to	$\frac{1}{x^3}$,	$\frac{1}{y^n}$.
$\frac{1}{\sqrt[4]{x^3}}$,	$\frac{1}{\sqrt[n]{x^m}}$,	equivalent to	$x^{-\frac{3}{4}}$,	$x^{-\frac{m}{n}}$.

The numerator of a fractional exponent indicates the power to which the quantity is to be raised, the denominator shows what root of that power is to be taken, and the sign of the exponent tells us whether the result is to be regarded as a factor, or as a divisor. Thus, the expression, $x^{-\frac{3}{4}}$, shows us that x is to be cubed, that the fourth root of this cube is to be extracted, and finally that reciprocal of the result is to be taken.

Demonstration of Principles.

125. Let n denote any whole number whatever, and assume

$$\sqrt[n]{a} \times \sqrt[n]{b} = p \quad . \quad . \quad . \quad . \quad (1)$$

Raising both members of (1) to the n^{th} power, remembering that $(\sqrt[n]{a})^n = a$, and $(\sqrt[n]{b})^n = b$, we have,

$$ab = p^n \quad . \quad . \quad . \quad . \quad (2)$$

Extracting the n^{th} root of both members of (2), we have,

$$\sqrt[n]{ab} = p \quad . \quad . \quad . \quad . \quad (3)$$

Things equal to the same thing are equal to each other; hence, equating the first members of (1) and (3), we have,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab} \quad . \quad . \quad . \quad (4)$$

Again, assume

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = q \quad . \quad . \quad . \quad . \quad (5)$$

Raising both members of (5) to the n^{th} power,

$$\frac{a}{b} = q^n \quad . \quad . \quad . \quad . \quad (6)$$

Extracting the n^{th} root of both members of (6),

$$\sqrt[n]{\frac{a}{b}} = q \quad . \quad . \quad . \quad . \quad (7)$$

Equating the first members of (5) and (7),

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad . \quad . \quad . \quad (8)$$

From the principle demonstrated in Art. 112, we have,

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} \quad . \quad . \quad . \quad . \quad (9)$$

From equations (4), (8), and (9), we have the following principles:

1°. *The product of the* n^{th} *roots of two quantities, is equal to the* n^{th} *root of their product, and the reverse.*

2°. *The quotient of the* n^{th} *roots of two quantities, is equal to the* n^{th} *root of their quotient, and the reverse.*

3°. The m^{th} *root of the* n^{th} *root of any quantity, is equal to the* mn^{th} *root of that quantity, and the reverse.*

These principles are used in the **transformation of radicals**, that is, in changing their forms, without affecting their values.

FIRST TRANSFORMATION. *To reduce a radical to its simplest form.*

126. A radical is in its **simplest form** when there is no factor under the sign which is a perfect power of the degree indicated.

Take the radical, $\sqrt{a^3x^4y^3}$:

Factoring the quantity under the radical signs, we have,

$$\sqrt{a^3x^4y^3} = \sqrt{a^2x^4y^2 \times ay}.$$

Hence, from principle 1°, we have,

$$\sqrt{a^3x^4y^3} = \sqrt{a^2x^4y^2} \times \sqrt{ay} = ax^2y\sqrt{ay}.$$

In a similar manner, other radicals may be simplified; hence, the following rule for reducing a radical to its simplest form:

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect power of the degree indicated. Extract the required root of this factor, and write the result as a factor without the radical sign, leaving the other factor under the sign.

Before pronouncing on the similarity of two radicals they should both be reduced to their simplest form.

EXAMPLES.

Reduce the following radicals to their simplest forms:

1. $\sqrt{48a^4x^2y} = \sqrt{16a^4x^2 \times 3y}$. *Ans.* $4a^2x\sqrt{3y}$.

2. $\sqrt{\frac{50a^2}{147b^2x}} = \sqrt{\frac{25a^2}{49b^2} \times \frac{2}{3x}}$. *Ans.* $\frac{5a}{7b}\sqrt{\frac{2}{3x}}$.

3. $\sqrt{(a+x)(a^2-x^2)} = \sqrt{(a+x)^2(a-x)}$.
Ans. $(a+x)\sqrt{a-x}$.

4. $\sqrt[3]{2a^4x+a^3x^2} = \sqrt[3]{a^3(2ax+x^2)}$.
Ans. $a\sqrt[3]{2ax+x^2}$.

5. $\sqrt[3]{\frac{24}{25}} + \sqrt[3]{\frac{5}{72}} + \sqrt[3]{\frac{192}{125}}$.
Ans. $2\sqrt[3]{\frac{3}{25}} + \frac{1}{2}\sqrt[3]{\frac{5}{9}} + \frac{4}{5}\sqrt[3]{3}$.

It will often be advantageous to multiply both terms of a fraction by such a quantity as will make the denominator a perfect power of the degree indicated, in which case, the factor remaining under the sign will be entire. Thus, in the 5th example,

$$\sqrt[3]{\frac{24}{25}} = \sqrt[3]{\frac{8 \times 3 \times 5}{125}} = \frac{2}{5}\sqrt[3]{15}; \text{ also, } \sqrt[3]{\frac{5}{72}} = \sqrt[3]{\frac{15}{216}} = \frac{1}{6}\sqrt[3]{15}.$$

6. $2\sqrt{\frac{3}{5}} + \sqrt{\frac{5}{3}}$. *Ans.* $\frac{2}{5}\sqrt{15} + \frac{1}{3}\sqrt{15}$.

7. $7\sqrt[3]{\frac{8}{3}} - 4\sqrt[4]{\frac{11}{4}}$. *Ans.* $\frac{14}{3}\sqrt[3]{9} - 2\sqrt[4]{44}$

8. $\sqrt{150} + \sqrt{1805} + \sqrt[3]{320}$.

Ans. $5\sqrt{6} + 19\sqrt{5} + 4\sqrt[3]{5}$.

9. $\sqrt[4]{768} + \sqrt[n]{a^{m+n}b^n} - \sqrt[5]{608}$.

Ans. $4\sqrt[4]{3} + ab\sqrt[n]{a^m} - 2\sqrt[5]{19}$.

10. $\sqrt{ax^2 - 6ax + 9a}$. *Ans.* $(x-3)\sqrt{a}$.

11. $(a^{4m}b^{2p}c^3)^{\frac{1}{2}} + (a^3 - a^2b)^{\frac{1}{2}}$.

Ans. $a^{2m}b^pc(c)^{\frac{1}{2}} + a(a-b)^{\frac{1}{2}}$.

12. $(a^4b^3c)^{\frac{1}{2}} + (a^5b^3)^{\frac{1}{3}}$ *Ans.* $a^2b(bc)^{\frac{1}{2}} + ab(a^2)^{\frac{1}{3}}$.

13. $\frac{1}{2}\sqrt{\frac{1}{2}} + \frac{1}{3}\sqrt[3]{\frac{1}{3}} + \frac{1}{4}\sqrt[4]{\frac{1}{4}}$. *Ans.* $\frac{1}{4}\sqrt{2} + \frac{1}{9}\sqrt[3]{9} + \frac{1}{8}\sqrt[4]{4}$.

14. $\sqrt{\dfrac{a^2b + a^2x}{b^3 - b^2x}}$. *Ans.* $\dfrac{a}{b}\sqrt{\dfrac{b+x}{b-x}}$.

SECOND TRANSFORMATION.—*To introduce a factor under the radical sign.*

127. Take the example, $4a\sqrt{2c}$:

Since $4a = \sqrt{16a^2}$, the given expression may be written,

$$4a\sqrt{2c} = \sqrt{16a^2} \times \sqrt{2c},$$

which, by principle (1), may be written,

$$4a\sqrt{2c} = \sqrt{16a^2 \times 2c} = \sqrt{32a^2c}.$$

In like manner, any factor without the radical sign may be introduced, as a factor, under the radical sign; hence, the following

RULE.

Raise the factor to the power indicated, and write it as a factor under the radical sign.

EXAMPLES.

Transform the following radicals by introducing the coefficients, as factors, under the radical signs:

1. $7\sqrt{3ax}$. *Ans.* $\sqrt{147ax}$.

2. $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$ *Ans.* $\sqrt[3]{\frac{12}{343}}$.

3. $7a\sqrt{2x} + \frac{1}{3}\sqrt[3]{12}$. *Ans.* $\sqrt{98a^2x} + \sqrt[3]{\frac{4}{9}}$.

4. $9\sqrt{3} + 3\sqrt[3]{3} + 2\sqrt[4]{8a^3}$
Ans. $\sqrt{243} + \sqrt[3]{81} + \sqrt[4]{128a^3}$.

5. $4\sqrt{2a} - 21a\sqrt{3a} - ax\sqrt[3]{x^2}$.
Ans. $\sqrt{32a} - \sqrt{1323a^3} - \sqrt[3]{a^3x^5}$.

6. $4\sqrt{\frac{7}{3}} - 5\sqrt{\frac{1}{2}} + 12\sqrt[3]{\frac{1}{4}}$.
Ans. $\sqrt{\frac{112}{3}} - \sqrt{\frac{25}{2}} + \sqrt[3]{432}$.

This transformation is used in finding the numerical values of radicals. Thus, it is easier to find the cube root of 432 than to find 12 times the cube root of $\frac{1}{4}$, to which it is equivalent.

THIRD TRANSFORMATION.—*To change the index of a radical.*

128. Take the radical, $\sqrt[3]{3a}$:

Since $3a$ is equal to $\sqrt{3a \times 3a}$, or $\sqrt{9a^2}$, we have,

$$\sqrt[3]{3a} = \sqrt[3]{\sqrt{9a^2}}.$$

But, from principle 3°, $\sqrt[3]{\sqrt{9a^2}} = \sqrt[6]{9a^2}$; hence,

$$\sqrt[3]{3a} = \sqrt[6]{9a^2}.$$

Here, a radical having an index 3 has been transformed to an equivalent radical, having an index 6.

In like manner,

$$\sqrt{2ax} = \sqrt[4]{4a^2x^2} = \sqrt[10]{(2ax)^5} = \sqrt[16]{(2ax)^8}, \text{ \&c.}$$

Again, we have from principle 3°,

$$\sqrt[4]{49a^2x^2} = \sqrt{\sqrt{49a^2x^2}}.$$

But, $\sqrt{49a^2x^2} = 7ax$; hence,

$$\sqrt[4]{49a^2x^2} = \sqrt{7ax}.$$

Since we may proceed in like manner in all similar cases, we have the following

Principles.

1°. *The index of any radical may be multiplied by any number, provided we raise the quantity under the* *radical sign to a power whose exponent is the same number.*

2°. *The index of a radical may be divided by any number, provided we extract that root of the quantity under the radical sign whose index is the same number.*

If a radical is expressed by means of a fractional exponent, we may proceed as follows:

$(a)^{\frac{1}{3}}$ is evidently equal to $(a)^{\frac{3}{9}}$, since $\frac{1}{3} = \frac{3}{9}$.

Also, $(a)^{\frac{3}{12}}$ is equal to $(a)^{\frac{1}{4}}$, since $\frac{3}{12}$ is equal to $\frac{1}{4}$; that is,

$$a^{\frac{1}{3}} = a^{\frac{3}{9}}; \text{ and } a^{\frac{3}{12}} = a^{\frac{1}{4}}.$$

or, by writing their equivalent expressions,

$$\sqrt[3]{a} = \sqrt[9]{a^3}, \qquad \sqrt[12]{a^3} = \sqrt[4]{a}.$$

These are the same results as obtained by the rule. From what precedes, we have the following

Principle.

3°. *Both terms of a fractional exponent may be multiplied, or divided, by the same quantity without changing the value of the radical.*

EXAMPLES.

Verify the following equations:

1. $\sqrt{3} + \sqrt[3]{4} + \sqrt[4]{5} = \sqrt[4]{9} + \sqrt[6]{16} + \sqrt[8]{25}.$

2. $\sqrt[4]{25} - \sqrt[6]{27} - \sqrt[8]{49} = \sqrt{5} - \sqrt{3} - \sqrt[4]{7}.$

3. $3\sqrt[4]{a^2-2ax+x^2} - 2\sqrt[6]{8a^3x^3} = 3\sqrt{a-x} - 2\sqrt{2ax}.$

4. $\sqrt{a-x}-\sqrt{a+x} = \sqrt[4]{a^2-2ax+x^2}-\sqrt[4]{a^2+2ax+x^2}$

5. $(ax)^{\frac{1}{2}} + (by)^{\frac{1}{3}} = (ax)^{\frac{2}{4}} + (by)^{\frac{2}{6}}.$

6. $(z)^{\frac{2}{4}} - (y)^{\frac{4}{6}} = (z)^{\frac{1}{2}} - (y)^{\frac{2}{3}}.$

FOURTH TRANSFORMATION. *To reduce radicals to a common index.*

129. Radicals may be reduced to a common index by means of the preceding principles. Let it be required to reduce the radicals, $\sqrt{a}$, $\sqrt[3]{b}$, and $\sqrt[4]{c}$, to equivalent ones having a common index: here, the least common multiple of the indices is 12; reducing each to the index 12, by the foregoing principles, we have,

$$\sqrt{a} = \sqrt[12]{a^6}, \quad \sqrt[3]{b} = \sqrt[12]{b^4}, \text{ and } \sqrt[4]{c} = \sqrt[12]{c^3}.$$

Since all other cases may be treated in the same way, we have the following

RULE.

Find the least common multiple of the indices, and reduce each radical to that index.

If the radicals are expressed by fractional exponents, we have simply to reduce these exponents to a common denominator.

EXAMPLES.

1. Reduce 2, $(3)^{\frac{1}{3}}$, $(a)^{\frac{1}{2}}$, and $(b)^{\frac{1}{4}}$, to a common index.

Ans. $(2^{12})^{\frac{1}{12}}$, $(3^4)^{\frac{1}{12}}$, $(a^6)^{\frac{1}{12}}$, and $(b^8)^{\frac{1}{12}}$.

2. Reduce $a^{\frac{1}{2}}$, b^2, $c^{\frac{2}{3}}$, and $d^{\frac{2}{3}}$, to a common index.

Ans. $(a^3)^{\frac{1}{6}}$, $(b^{12})^{\frac{1}{6}}$, $(c^4)^{\frac{1}{6}}$, and $(d^4)^{\frac{1}{6}}$.

3. Reduce $\sqrt{a+x}$, $\sqrt[3]{a-x}$, and $\sqrt[4]{a^2-x^2}$, to a common index.

Ans. $\sqrt[12]{(a+x)^6}$, $\sqrt[12]{(a-x)^4}$, and $\sqrt[12]{(a^2-x^2)^3}$.

4. Reduce $\sqrt{\frac{2}{3}}$, $\sqrt[3]{2}$, and $5\sqrt{3}$, to a common index.

Ans. $\sqrt[6]{\frac{8}{27}}$, $\sqrt[6]{4}$, and $5\sqrt[6]{27}$.

5. Reduce ax, $(bx)^{\frac{1}{2}}$, $(cx)^{\frac{1}{3}}$, and $(dx)^{\frac{1}{4}}$, to a common index.

Ans. $(a^{12}x^{12})^{\frac{1}{12}}$, $(b^6x^6)^{\frac{1}{12}}$, $(c^4x^4)^{\frac{1}{12}}$, and $(d^3x^3)^{\frac{1}{12}}$.

6. Reduce cx^2, $(dx^3)^{\frac{1}{4}}$, and $(z^4)^{\frac{1}{2}}$, to a common index.

Ans. $(cx^2)^{\frac{4}{4}}$, $(dx^3)^{\frac{1}{4}}$, and $(z^4)^{\frac{2}{4}}$.

7. Reduce $\sqrt[3]{7}$, $\sqrt{10}$, and $\sqrt[6]{\frac{161}{49}}$, to a common index. *Ans.* $\sqrt[6]{49}$, $\sqrt[6]{1000}$, and $\sqrt[6]{\frac{161}{49}}$.

8. Reduce $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{16}{49}}$, and $\sqrt[6]{1331}$, to a common index. *Ans.* $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{4}{7}}$, and $\sqrt{11}$.

II. FUNDAMENTAL OPERATIONS ON RADICALS.

1°. Addition of Radicals.

130. Radicals cannot be added unless they are *similar*. To determine when they are similar, we must reduce them to their simplest form; then, if their radical parts are the same, they will be similar, and if we regard the common radical part as a unit, we shall have the following rule for finding their sum:

RULE.

Reduce the radicals to their simplest forms; then, if they are similar, add the coefficients for a new coefficient, and write the sum before the common radical part.

EXAMPLES.

Find the sums ot the following groups of radicals:

1. $\sqrt{18}$, $\sqrt{32}$, $\sqrt{50}$, and $\sqrt{72}$.

   Ans. $3\sqrt{2} + 4\sqrt{2} + 5\sqrt{2} + 6\sqrt{2} = 18\sqrt{2}$.

2. $2\sqrt{8}$; $3\sqrt{50}$, and $6\sqrt{18}$. *Ans.* $37\sqrt{2}$.

3. $\sqrt{\frac{3}{5}}$, $\sqrt{\frac{1}{15}}$, and $\sqrt{\frac{15}{49}}$. *Ans.* $\frac{43}{105}\sqrt{15}$.

4. $\frac{2}{3}\sqrt[3]{\frac{2}{9}}$, $\frac{1}{6}\sqrt[3]{\frac{1}{36}}$, and $\frac{3}{5}\sqrt[3]{\frac{3}{32}}$. *Ans.* $\frac{2}{5}\sqrt[3]{6}$.

5. $x\sqrt{12a^4x}$, $2a^2\sqrt{27x^3}$, $3a\sqrt{48a^2x^3}$, and $5a^2x\sqrt{3x}$.

   Ans. $25a^2x\sqrt{3x}$.

6. $\sqrt[3]{54a^{n+6}b^3}$, $a\sqrt[3]{16a^{n-3}b^6}$, and $\sqrt[3]{2a^{4n+9}}$.

   Ans. $(3a^2b + 2b^2 + a^{n+3})\sqrt[3]{2a^n}$.

7. $6\sqrt[6]{4a^2}$, $2\sqrt[3]{2a}$, and $\sqrt[9]{8a^3}$. *Ans.* $9\sqrt[3]{2a}$.

8. $2\sqrt{3}$, $\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$, and $2\sqrt{\frac{3}{16}}$. *Ans.* $\frac{31}{2}\sqrt{3}$.

9. $3b\sqrt[3]{2a^5b^2}$, $7\sqrt[3]{2a^5b^5}$, and $8a\sqrt[3]{2a^2b^5}$.

   Ans. $18ab\sqrt[3]{2a^2b^2}$.

10. $\frac{ab}{b-c}$, and $\sqrt{\frac{a^2b^2}{(b-c)^2} - \frac{a^2b}{b-c}}$

Ans. $\frac{a(b+\sqrt{bc})}{b-c}$.

2°. Subtraction of Radicals.

131. We cannot subtract one radical from another unless the two are similar. In that case, we have the following

RULE.

Reduce the radicals to their simplest forms; then, if they are similar, subtract the coefficient of the subtrahend from that of the minuend, and write the remainder before the common radical part.

EXAMPLES.

1. From $\sqrt{320}$, subtract $\sqrt{80}$.

Ans. $8\sqrt{5} - 4\sqrt{5} = 4\sqrt{5}$.

2. From $b\sqrt[3]{27a^6b}$, subtract $\sqrt[3]{216a^6b^4}$.

Ans. $-3a^2b\sqrt[3]{b}$.

3. From $\sqrt{a^3+2a^2b+ab^2}$, subtract $\sqrt{a^3-2a^2b+ab^2}$.

Ans. $2b\sqrt{a}$.

4. From $\frac{2}{3}\sqrt[3]{\frac{2}{9}} + \frac{3}{5}\sqrt[3]{\frac{3}{32}}$, subtract $\frac{1}{6}\sqrt[3]{\frac{1}{36}}$.

Ans. $\frac{31}{90}\sqrt[3]{6}$.

5. From $\sqrt{289a^2b}$, subtract $\sqrt{144a^2b}$. *Ans.* $5a\sqrt{b}$.

6. From $2\sqrt{8a^3}+5\sqrt{72a^3}$, subtract $7a\sqrt{18a}+\sqrt{50ab^2}$.

Ans. $(13a-5b)\sqrt{2a}$.

7. From $(a-x)\sqrt{a^2-x^2}$, subtract $\sqrt{\dfrac{a+x}{a-x}}$.

Ans. $\left(a-x-\dfrac{1}{a-x}\right)\sqrt{a^2-x^2}$.

8. From $\sqrt[3]{81}+\sqrt[3]{192}$, subtract $\sqrt[3]{512}$.

Ans. $7\sqrt[3]{3}-8$.

3°. Multiplication of Radicals.

132. Since two radicals can always be reduced to a common index, we may take $a\sqrt[n]{b}$, and $c\sqrt[n]{d}$, to represent any two radicals whatever. The indicated product is,

$$a\sqrt[n]{b} \times c\sqrt[n]{d}.$$

We may change the order of the factors without changing the value of the product; hence, may write the product under the form,

$$ac\sqrt[n]{b} \times \sqrt[n]{d}.$$

But, from principle 1°, $\sqrt[n]{b} \times \sqrt[n]{d} = \sqrt[n]{bd}$; hence,

$$a\sqrt[n]{b} \times c\sqrt[n]{d} = ac\sqrt[n]{bd};$$

whence the following

RULE.

Reduce the radicals to a common index; then multiply the coefficients together for a new coefficient, and the quantities under the radical signs for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

Perform the following indicated multiplications:

1. $3\sqrt{8} \times 4\sqrt{48}$. *Ans.* $12\sqrt{384} = 96\sqrt{6}$.

2. $\frac{7}{3}\sqrt[3]{\frac{2}{3}} \times \frac{13}{4}\sqrt[3]{\frac{3}{4}}$. *Ans.* $\frac{91}{12}\sqrt[3]{\frac{1}{2}} = \frac{91}{24}\sqrt[3]{4}$.

3. $4\sqrt{12} \times 3\sqrt{2}$. *Ans.* $12\sqrt{24} = 24\sqrt{6}$.

4. $\frac{1}{3}\sqrt[3]{4} \times \frac{3}{4}\sqrt[3]{12}$. *Ans.* $\frac{1}{4}\sqrt[3]{48} = \frac{1}{2}\sqrt[3]{6}$.

5. $5a\sqrt{ax} \times \frac{5}{2}\sqrt{bx}$. *Ans.* $\frac{25}{2}a\sqrt{abx^2} = \frac{25ax}{2}\sqrt{ab}$.

6. $\sqrt{2ab^3} \times \sqrt{8a^3b}$. *Ans.* $\sqrt{16a^4b^4} = 4a^2b^2$.

7. $\sqrt{8} \times \sqrt[3]{5}$.

Ans. $\sqrt[6]{512} \times \sqrt[6]{25} = \sqrt[6]{12800} = 2\sqrt[6]{200}$.

8. $\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{3}{4}}$.

Ans. $\sqrt[6]{\frac{1}{4}} \times \sqrt[6]{\frac{27}{64}} = \sqrt[6]{\frac{27}{256}} = \frac{1}{2}\sqrt[6]{\frac{27}{4}}$.

9. $\frac{1}{8}\sqrt[3]{\frac{1}{8}} \times \frac{1}{6}\sqrt[3]{\frac{1}{6}}$. *Ans.* $\frac{1}{48}\sqrt[3]{\frac{1}{48}} = \frac{1}{96}\sqrt[3]{\frac{1}{6}}$.

By combining the above rule with that for the multiplication of polynomials, complicated radical expressions may be multiplied together.

10. Multiply $\sqrt[3]{x} + 2\sqrt[6]{x} + 4$, by $\sqrt[3]{x} + 2\sqrt[6]{x}$.

FIRST OPERATION.	SECOND OPERATION.
$\sqrt[3]{x} + 2\sqrt[6]{x} + 4$	$x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 4$
$\sqrt[3]{x} + 2\sqrt[6]{x}$	$x^{\frac{1}{3}} + 2x^{\frac{1}{6}}$
$\sqrt[3]{x^2} + 2\sqrt[6]{x^3} + 4\sqrt[3]{x}$	$x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 4x^{\frac{1}{3}}$
$+ 2\sqrt[6]{x^3} + 4\sqrt[3]{x} + 8\sqrt[6]{x}$	$+ 2x^{\frac{1}{2}} + 4x^{\frac{1}{3}} + 8x^{\frac{1}{6}}$
$\sqrt[3]{x^2} + 4\sqrt{x} + 8\sqrt[3]{x} + 8\sqrt[6]{x}$	$x^{\frac{2}{3}} + 4x^{\frac{1}{2}} + 8x^{\frac{1}{3}} + 8x^{\frac{1}{6}}$

These results are identical, but the second has been obtained by following the ordinary rule for exponents; hence, we conclude that the rule for multiplication is the same whether the exponents are entire or fractional.

11. $(a^{\frac{3}{4}} + a^{\frac{1}{2}}x^{\frac{1}{2}} + a^{\frac{1}{4}}x + x^{\frac{3}{2}}) \times (a^{\frac{1}{4}} - x^{\frac{1}{2}})$.

Ans. $a - x^2$.

12. $\left(x + \frac{p}{2} + \sqrt{q + \frac{p^2}{4}}\right) \times \left(x + \frac{p}{2} - \sqrt{q + \frac{p^2}{4}}\right)$.

Ans. $x^2 + px - q$.

13. $\left(\frac{x}{b}\sqrt{\frac{a}{b}} + \sqrt{\frac{c}{d}}\right) \times \left(\frac{x}{b}\sqrt{\frac{a}{b}} - \sqrt{\frac{c}{d}}\right)$

Ans. $\frac{ax^2}{b^3} - \frac{c}{d}$.

14. $\left(\sqrt[3]{a^{-\frac{1}{2}}} + \sqrt[6]{a^{\frac{1}{2}}b}\right) \times \left(\sqrt[3]{a^{-\frac{1}{2}}} - \sqrt[6]{a^{\frac{1}{2}}b}\right)$.

Ans. $\sqrt[3]{a^{-1}} - \sqrt[3]{a^{\frac{1}{2}}b}$.

15. $\left(\frac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}\right) \times \left(\frac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}\right).$

Ans. $\frac{a^2}{4}+\frac{c}{2}\sqrt{a^2-c^2}.$

16. $\left(8x^{\frac{3}{4}}+2x^{\frac{1}{2}}y+\frac{1}{2}x^{\frac{1}{4}}y^2+\frac{1}{8}y^3\right) \times \left(2x^{\frac{1}{4}}-\frac{y}{2}\right).$

Ans. $16x-\frac{y^4}{16}.$

4°. Division of Radicals.

133. Let $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ represent any two radicals, after having been reduced to a common index. The quotient of the first by the second may be represented as follows:

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\frac{a}{c}\times\frac{\sqrt[n]{b}}{\sqrt[n]{d}}.$$

But, from principle 2°, $\frac{\sqrt[n]{b}}{\sqrt[n]{d}}=\sqrt[n]{\frac{b}{d}}$; hence,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\frac{a}{c}\sqrt[n]{\frac{b}{d}}.$$

Hence, the following

RULE.

Reduce the radicals to a common index; then divide the coefficient of the dividend by that of the divisor for a new coefficient, and the quantity under the radical sign in the dividend by that in the divisor for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

Perform the following indicated divisions:

1. $\frac{7}{3}\sqrt[3]{\frac{2}{3}} \div \frac{13}{4}\sqrt[3]{\frac{1}{5}}$. *Ans.* $\frac{28}{39}\sqrt[3]{\frac{10}{3}}$.

2. $\frac{1}{4}\sqrt{\frac{2}{5}} \div \frac{3}{7}\sqrt{\frac{5}{2}}$. *Ans.* $\frac{7}{12}\sqrt{\frac{4}{25}} = \frac{7}{30}$.

3. $\frac{1}{2}\sqrt{2ax} \div \frac{3}{4}\sqrt{2bx}$. *Ans.* $\frac{2}{3}\sqrt{\frac{a}{b}}$.

4. $\frac{1}{2}\sqrt{\frac{2}{3}} \div \frac{1}{3}\sqrt[3]{\frac{1}{3}}$. *Ans.* $\frac{1}{2}\sqrt[6]{\frac{8}{27}} \div \frac{1}{3}\sqrt[6]{\frac{1}{9}} = \frac{3}{2}\sqrt[6]{\frac{8}{3}}$.

5. $2\sqrt{2ax} \div \sqrt[3]{4bx^2}$.

Ans. $2\sqrt[6]{8a^3x^3} \div \sqrt[6]{16b^2x^4} = 2\sqrt[6]{\frac{a^3}{2b^2x}}$.

By combining the above rule with that for the division of polynomials, any complicated radical expression may be divided by another.

9. Divide $x + \sqrt{xy} + y$, by $\sqrt{x} + \sqrt[4]{xy} + \sqrt{y}$.

FIRST OPERATION.

$$\begin{array}{l|l}
x + \sqrt{xy} + y & \sqrt{x} + \sqrt[4]{xy} + \sqrt{y} \\ \cline{2-2}
x + \sqrt[4]{x^3y} + \sqrt{xy} & \sqrt{x} - \sqrt[4]{xy} + \sqrt{y} \\ \cline{1-1}
-\sqrt[4]{x^3y} + y & \\
-\sqrt[4]{x^3y} - \sqrt{xy} - \sqrt[4]{xy^3} & \\ \hline
\qquad \sqrt{xy} + \sqrt[4]{xy^3} + y & \\
\qquad \sqrt{xy} + \sqrt[4]{xy^3} + y & \\ \hline
\qquad\qquad 0 &
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l|l}
x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y & x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\ \cline{2-2}
x + x^{\frac{3}{4}}y^{\frac{1}{4}} + x^{\frac{1}{2}}y^{\frac{1}{2}} & x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}} \\ \cline{1-1}
\quad - x^{\frac{3}{4}}y^{\frac{1}{4}} + y & \\
\quad - x^{\frac{3}{4}}y^{\frac{1}{4}} - x^{\frac{1}{2}}y^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{3}{4}} & \\ \cline{1-1}
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y & \\
\qquad\qquad x^{\frac{1}{2}}y^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{3}{4}} + y & \\ \cline{1-1}
\qquad\qquad\qquad\quad 0 &
\end{array}$$

These results are identical; but the second one has been obtained by following the ordinary rule for exponents. Hence, we conclude that the operation for division is the same, whether the exponents are entire or fractional.

7. $\left(16x - \dfrac{y^4}{16}\right) \div \left(2x^{\frac{1}{4}} - \dfrac{y}{2}\right)$.

Ans. $8x^{\frac{3}{4}} + 2x^{\frac{1}{2}}y + \frac{1}{2}x^{\frac{1}{4}}y^2 + \frac{1}{8}y^3$.

5°. Reduction of Radicals.

134. It is often desirable to transform radical expressions of the form $\dfrac{a}{\sqrt{b} + \sqrt{c}}$, and $\dfrac{a}{\sqrt{b} - \sqrt{c}}$, into equivalent expressions, in which the denominator is rational, that is, which does not contain any radical.

The first form may be thus transformed, by multiplying both terms by $\sqrt{b} - \sqrt{c}$; and the second, by multiplying both terms by $\sqrt{b} + \sqrt{c}$, giving

$$\frac{a\sqrt{b} - a\sqrt{c}}{b - c}, \text{ and } \frac{a\sqrt{b} + a\sqrt{c}}{b - c}.$$

If only one term of the denominator contains a radical, the same rule will hold good.

EXAMPLES.

Render the denominators of the following fractions rational:

1. $\frac{2}{\sqrt{3}+2}$. *Ans.* $\frac{2\sqrt{3}-4}{3-4}$.

2. $\frac{3}{\sqrt{2}-\sqrt{3}}$. *Ans.* $\frac{3\sqrt{2}+3\sqrt{3}}{2-3}$.

3. $\frac{4}{11-2\sqrt{3}}$ *Ans.* $\frac{44+8\sqrt{3}}{109}=\frac{44}{109}+\frac{8}{109}\sqrt{3}$.

4. $\frac{3}{8+\sqrt{2}}$ *Ans.* $\frac{24-3\sqrt{2}}{62}=\frac{12}{31}-\frac{3}{62}\sqrt{2}$.

5. $(\sqrt{3}-\sqrt{2}) \div (\sqrt{2}+1)$.
Ans. $\sqrt{2}-\sqrt{3}+\sqrt{6}-2$.

6. $4 \div (\sqrt{5}+1)$. *Ans.* $\sqrt{5}-1$.

7. $(\sqrt{a+x}+\sqrt{a-x}) \div (\sqrt{a+x}-\sqrt{a-x})$.
Ans. $\frac{a}{x}+\sqrt{\frac{a^2}{x^2}-1}$.

135. In the solution of certain equations, it often becomes necessary to extract the square root of expressions of the form, $a+\sqrt{b}$, and $a-\sqrt{b}$. In some cases, this operation may be performed, in other cases it cannot be performed. To investigate a rule for determining when the operation can be performed, and the manner of performing it, assume

$$\sqrt{a+\sqrt{b}} = x+y \quad . \quad . \quad . \quad . \quad (1)$$

$$\sqrt{a-\sqrt{b}} = x-y \quad . \quad . \quad . \quad . \quad (2)$$

Squaring both members of (1) and (2),

$$a + \sqrt{b} = x^2 + 2xy + y^2 \quad . \quad . \quad (3)$$

$$a - \sqrt{b} = x^2 - 2xy + y^2 \quad . \quad . \quad (4)$$

Adding (3) and (4), and omitting the common factor 2,

$$a = x^2 + y^2 \quad . \quad . \quad . \quad . \quad (5)$$

Multiplying (1) by (2),

$$\sqrt{a^2 - b} = x^2 - y^2 \quad . \quad . \quad . \quad . \quad (6)$$

Adding and subtracting (5) and (6),

$$x^2 = \frac{a + \sqrt{a^2 - b}}{2} \quad . \quad . \quad (7)$$

$$y^2 = \frac{a - \sqrt{a^2 - b}}{2} \quad . \quad . \quad (8)$$

Now, if $a^2 - b$ is a perfect square, its root may be represented by c. Substituting in (7) and (8), and extracting the square root of each member of both equations, (axiom 5), we have,

$$x = \sqrt{\frac{a + c}{2}}, \text{ and } y = \sqrt{\frac{a - c}{2}}.$$

These values, substituted in (1) and (2), give,

$$\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + c}{2}} + \sqrt{\frac{a - c}{2}} \quad . \quad . \quad (9)$$

$$\sqrt{a - \sqrt{b}} = \sqrt{\frac{a + c}{2}} - \sqrt{\frac{a - c}{2}} \quad . \quad . \quad (10)$$

The square root of the given quantities may be extracted when $a^2 - b$ is a perfect square, and the roots may be obtained, by substitution, from (9) and (10).

EXAMPLES.

1. Required the square root of $14 + 6\sqrt{5} = 14 + \sqrt{180}$.

Here, $a = 14$, $b = 180$, and $c = \sqrt{196 - 180} = 4$: hence,

$$\sqrt{14 + 6\sqrt{5}} = \sqrt{\frac{14+4}{2}} + \sqrt{\frac{14-4}{2}} = 3 + \sqrt{5}.$$

2. Required the square root of $18 - 2\sqrt{77}$.

Here, $a = 18$, $b = 308$, and $c = \sqrt{324 - 308} = 4$; hence,

$$\sqrt{18 - 2\sqrt{77}} = \sqrt{\frac{22}{2}} - \sqrt{\frac{14}{2}} = \sqrt{11} - \sqrt{7}.$$

3. Required the square root of $94 + 42\sqrt{5}$.

Ans. $7 + 3\sqrt{5}$.

4. Required the square root of $28 + 10\sqrt{3}$.

Ans. $5 + \sqrt{3}$.

6°. Operations on Imaginary Quantities.

136. An **imaginary quantity** has been defined to be an indicated even root of a negative quantity.

The rule deduced for multiplying radicals requires some modification, when applied to imaginary quantities. By the rule already deduced, the product of $\sqrt{-4}$ by $\sqrt{-3}$ would be equal to $\sqrt{12}$; whereas, the true product is $-\sqrt{12}$, as will be shown hereafter.

Every imaginary quantity of the second degree can, by principle 1°, (Art. 125), be resolved into two factors, one of which is $\sqrt{-1}$; the other factor may be either rational or irrational. Thus,

$$\sqrt{-4} = 2\sqrt{-1},\ \sqrt{-3} = \sqrt{3} \times \sqrt{-1},\ \sqrt{-a^2} = a\sqrt{-1}.$$

The factor, $\sqrt{-1}$, is called the **imaginary factor**, and the other one is called its **coefficient**. Thus, in the expression, $\sqrt{3} \times \sqrt{-1}$, the factor $\sqrt{3}$ is the coefficient of the imaginary factor $\sqrt{-1}$.

When several imaginary factors are to be multiplied together, we first reduce them to the form, $a\sqrt{-1}$. We can then multiply together the coefficients of the imaginary factor by known rules. It remains to deduce a rule for multiplying together the imaginary factors, or what is the same thing, for raising the imaginary factor to a power whose exponent is equal to the number of factors.

The first power of $\sqrt{-1}$, is $\sqrt{-1}$; the second power, by the definition of square root, is -1; the third power, is the product of the first and second powers, or $-1 \times \sqrt{-1} = -\sqrt{-1}$; the fourth power, is the square of the second power, or $+1$; the fifth, is the product of the first and fourth, that is, it is the same as the first; the sixth, is the same as the second; the seventh, the same as the third; the eighth, the same as the fourth; the ninth, again, the same as the first; and so on indefinitely, as shown in the table, *n* being any whole number.

$$\begin{array}{llllll} (\sqrt{-1})^1 & = & \sqrt{-1}. & (\sqrt{-1})^{4n+1} & = & \sqrt{-1}. \\ (\sqrt{-1})^2 & = & -1. & (\sqrt{-1})^{4n+2} & = & -1. \\ (\sqrt{-1})^3 & = & -\sqrt{-1}. & (\sqrt{-1})^{4n+3} & = & -\sqrt{-1}. \\ (\sqrt{-1})^4 & = & +1. & (\sqrt{-1})^{4n} & = & 1. \end{array}$$

To show the use of this table, let it be required to find the continued product of $\sqrt{-4}$, $\sqrt{-3}$, $\sqrt{-2}$, $\sqrt{-7}$, and $\sqrt{-8}$. Reducing these expressions to the proper form, and indicating the multiplication, we have,

$$2\sqrt{-1} \times \sqrt{3}\sqrt{-1} \times \sqrt{2}\sqrt{-1} \times \sqrt{7}\sqrt{-1} \times 2\sqrt{2}\sqrt{-1}.$$

Changing the order of the factors,

$$(2 \times \sqrt{3} \times \sqrt{2} \times \sqrt{7} \times 2\sqrt{2})\,(\sqrt{-1})^5.$$

Hence, the product is equal to, $8\sqrt{21} \times \sqrt{-1} = 8\sqrt{-21}$.

EXAMPLES.

Perform the multiplications indicated below:

1. $\sqrt{-a^2} \times \sqrt{-b^2}$. *Ans.* $a \times b(\sqrt{-1})^2 = -ab$.

2. $\sqrt{-a^2} \times \sqrt{-b^2} \times \sqrt{-c^2}$.
Ans. $abc(\sqrt{-1})^3, = -abc\sqrt{-1}$.

3. $\sqrt{-a^2} \times \sqrt{-b^2} \times \sqrt{-c^2} \times \sqrt{-d^2}$.
Ans. $abcd(\sqrt{-1})^4 = abcd$.

4. $(4 + \sqrt{-2}) \times (3 - \sqrt{-2})$. *Ans.* $14 - \sqrt{-2}$.

5. $(2 - \sqrt{-2}) \times (2 - \sqrt{-2})$. *Ans.* $2 - 4\sqrt{-2}$.

6. $(3 - \sqrt{-2}) \times (3 + \sqrt{-2})$. *Ans.* 11.

From what precedes, it follows that the only radical parts of any power of an expression of the form, $a \pm b\sqrt{-1}$, will be of the form $c\sqrt{-1}$.

Properties of Imaginary Quantities.

138. 1°. *A quantity of the form, $a\sqrt{-1}$, cannot be equal to the sum of a rational quantity and a quantity of the form, $b\sqrt{-1}$*

For, if so, let us have the equality,

$$a\sqrt{-1} = x + b\sqrt{-1};$$

squaring both members, we have,

$$-a^2 = x^2 + 2bx\sqrt{-1} - b^2;$$

transposing, and dividing by $2bx$,

$$\sqrt{-1} = \frac{b^2 - a^2 - x^2}{2bx},$$

an equation which is manifestly absurd, for the first member is imaginary, and the second real, and no imaginary quantity can be equal to a real quantity; hence, the hypothesis is absurd; and, consequently, the principle enunciated is true.

In the same way, it may be shown that no radical of the second degree can be equal to an entire quantity plus a radical of the second degree.

2°. *If, $a + b\sqrt{-1} = x + y\sqrt{-1}$, then $a = x$, and $b = y$*

For, by transposition, we have,

$$b\sqrt{-1} = (x - a) + y\sqrt{-1};$$

but from the preceding principle, this equation can only be true when $x - a = 0$, or $x = a$; making this supposition, and dividing both members of the given equation by $\sqrt{-1}$, we have $b = y$, which was to be shown.

In the same way, it may be shown that, when $a + \sqrt{b} = x + \sqrt{y}$, we have, $a = x$, and $b = y$: that is, in all equations of this form, *the rational and radical parts, in each member, are respectively equal to each other.*

3°. *The product of two factors, of the forms.* $x - (a + b\sqrt{-1})$, *and* $x - (a - b\sqrt{-1})$, *is positive for all real values of* x:

For, performing the multiplication, we find the product equal to,

$$x^2 - 2ax + a^2 + b^2,$$

which can be written under the form,

$$(x - a)^2 + b^2.$$

Now, whatever may be the value of x, the part $(x - a)^2$ will be positive, since it is a square; b^2 is also positive; hence, their sum, or the required product, is also positive, which was to be proved.

III. SOLUTION OF RADICAL EQUATIONS.

139. Radical equations are equations containing radical quantities. No fixed rules can be given for solving such equations. The methods of proceeding will be best illustrated by examples.

EXAMPLES.

1. Given $\sqrt{x+16} = 2 + \sqrt{x}$, to find x.

Squaring,

$$x + 16 = 4 + 4\sqrt{x} + x;$$

transposing and reducing,

$$\sqrt{x} = 3;$$

squaring,

$$x = 9.$$

2. Given $1 - \sqrt{1-x} = n(1 + \sqrt{1-x})$, to find x.

Reducing,

$$1 - n = (n + 1)\sqrt{1 - x};$$

squaring,

$$1 - 2n + n^2 = (n^2 + 2n + 1)(1 - x);$$

whence,

$$x = \frac{4n}{(1 + n)^2}.$$

3. Given $\sqrt{a+x} - \sqrt{a-x} = \sqrt{ax}$, to find x.

Squaring and reducing,

$$2a - ax = 2\sqrt{a^2 - x^2},$$

squaring again,

$$4a^2 - 4a^2x + a^2x^2 = 4a^2 - 4x^2;$$

reducing, and dividing by x,

$$(a^2 + 4)x = 4a^2;$$

whence,

$$x = \frac{4a^2}{a^2 + 4}.$$

4. Given $\dfrac{x-1}{\sqrt{x}+1} = 4 + \dfrac{\sqrt{x}-1}{2}$, to find x.

Reducing,

$$\sqrt{x} - 1 = 4 + \frac{\sqrt{x} - 1}{2};$$

or,

$$\sqrt{x} - 1 = 8.$$

Transposing,

$$\sqrt{x} = 9: \quad \therefore \quad x = 81.$$

5. Given $\dfrac{x}{\sqrt{a^2 + x^2}} = \dfrac{c - x}{\sqrt{b^2 + (c-x)^2}}$, to find x.

Squaring,

$$\frac{x^2}{a^2 + x^2} = \frac{(c - x)^2}{b^2 + (c - x)^2};$$

clearing of fractions,

$$b^2x^2 + x^2(c - x)^2 = a^2(c - x)^2 + x^2(c - x)^2;$$

reducing,

$$b^2x^2 = a^2(c - x)^2;$$

extracting square root,

$$bx = a(c - x).$$

$$\therefore \; x = \frac{ac}{a + b}.$$

6. Given $\dfrac{x - a}{\sqrt{x} + \sqrt{a}} = \dfrac{\sqrt{x} - \sqrt{a}}{3} + 2\sqrt{a}$, to find x.

Reducing,

$$\sqrt{x} - \sqrt{a} = \frac{\sqrt{x} - \sqrt{a}}{3} + 2\sqrt{a};$$

whence,

$$\sqrt{x} - \sqrt{a} = 3\sqrt{a};$$

or,

$$\sqrt{x} = 4\sqrt{a}.$$

$$\therefore \; x = 16a.$$

7. Given $8\sqrt{3x} + \dfrac{81(3 + 4\sqrt{3x})}{16x - 3} = 16x + 3$, to find x.

Transposing,

$$16x - 8\sqrt{3}\sqrt{x} + 3 = \frac{81(3 + 4\sqrt{3}\sqrt{x})}{(4\sqrt{x} - \sqrt{3})(4\sqrt{x} + \sqrt{3})};$$

factoring,

$$(4\sqrt{x} - \sqrt{3})^2 = \frac{81\sqrt{3}(\sqrt{3} + 4\sqrt{x})}{(4\sqrt{x} - \sqrt{3})(4\sqrt{x} + \sqrt{3})}$$

reducing,

$$(4\sqrt{x} - \sqrt{3})^3 = 81\sqrt{3} = 27\sqrt{27};$$

extracting cube root,

$$4\sqrt{x} - \sqrt{3} = 3\sqrt{3};$$

transposing,

$$4\sqrt{x} = 3\sqrt{3} + \sqrt{3} = 4\sqrt{3};$$

dividing by 4, and squaring,

$$x = \frac{16 \times 3}{16} = 3$$

Solve the following equations:

8. $\sqrt{x} = 1 + \sqrt{x - 9}.$ *Ans.* $x = 25.$

9. $\sqrt{x} + \sqrt{x - 3} = 3.$ *Ans.* $x = 4.$

10. $\sqrt{x} - \sqrt{2} = \sqrt{x - 2}.$ *Ans.* $x = 2.$

11. $\sqrt[3]{4x + 3} = 3.$ *Ans.* $x = 6.$

12. $\sqrt{5x + 4} = 2 + \sqrt{3x}.$ *Ans.* $x = 12.$

13. $2\sqrt{x} - \sqrt{a} = 2\sqrt{x - a}.$ *Ans.* $x = \frac{25a}{16}.$

14. $a + x = \sqrt{x^2 + 5x - a}.$ *Ans.* $x = \frac{a^2 + a}{5 - 2a}.$

15. $\sqrt{a - x} = \frac{a}{\sqrt{a - x}} - x.$ *Ans.* $x = a - 1.$

16. $\frac{\sqrt{x} - 2}{3} + 3 = \frac{x - 4}{\sqrt{x} + 2}.$ *Ans.* $x = 42\frac{1}{4}.$

17. $x - \sqrt{a^2 + x\sqrt{x^2 - 1}} = a.$ *Ans.* $x = \frac{4a^2 + 1}{4a}.$

18. $\sqrt{x + a} = \sqrt{a} + \sqrt{x - a}.$ *Ans.* $x = \frac{5a}{4}.$

19. $\sqrt{x} - \sqrt{a-x} = \dfrac{\sqrt{x} + \sqrt{a-x}}{2}$. *Ans.* $x = \dfrac{9a}{10}$.

20. $\sqrt[3]{1+x} + \sqrt[3]{1-x} = \sqrt[3]{2}$. *Ans.* $x = \pm 1$.

21. $\dfrac{ax-1}{\sqrt{ax}+1} = 4 + \dfrac{\sqrt{ax}-1}{2}$. *Ans.* $x = \dfrac{81}{a}$.

22. $\dfrac{1}{x} + \dfrac{1}{a} = \sqrt{\dfrac{1}{a^2} + \sqrt{\dfrac{4}{b^2x^2} + \dfrac{1}{x^4}}}$.

Ans. $x = \dfrac{ab^2}{a^2 - b^2}$

23. $a + x + \sqrt{a^2 + bx + x^2} = b$. *Ans.* $x = \dfrac{b^2 - 2ab}{3b - 2a}$.

24. $\dfrac{x-9}{\sqrt{x}+3} + \dfrac{x-4}{\sqrt{x}-2} = \dfrac{4(x-16)}{\sqrt{x}+4}$. *Ans.* $x = 56\frac{1}{4}$.

CHAPTER IX.

EQUATIONS OF THE SECOND DEGREE.

I. EQUATIONS CONTAINING BUT ONE UNKNOWN QUANTITY.

Reduction to Particular Form.

140. Any equation of the second degree, containing but one unknown quantity, can always be reduced to the form of

$$x^2 + 2px = q.$$

Take the equation,

$$\frac{x^2}{2} - \frac{6x}{4} + 5 = 6 - \frac{4x^2}{6} + \frac{(2x + 1)^2}{3};$$

clearing of fractions, and performing indicated operations, we have,

$$6x^2 - 18x + 60 = 72 - 8x^2 + 16x^2 + 16x + 4;$$

transposing the unknown terms to the first member, and the known terms to the second member, we have,

$$6x^3 + 8x^2 - 16x^2 - 18x - 16x = 72 + 4 - 60;$$

factoring and reducing, we have,

$$-2x^2 - 34x = 16;$$

dividing by the coefficient of x^2, that is, by -2,

$$x^2 + 17x = -8;$$

which is of the required form, $2p$, in this case, being equal to 17, and q being equal to -8.

All other equations of the same kind may be treated in the same manner; hence, we have the following rule for reducing equations of the second degree, containing but one unknown quantity, to the form

$$x^2 + 2px = q:$$

RULE.

I. Clear the equation of fractions, and perform all the indicated operations.

II. Transpose all the unknown terms to the first member, and all the known terms to the second member.

III. Reduce all the terms containing the square of the unknown quantity to a single term, one factor of which is the square of the unknown quantity; reduce, also, all the terms containing the first power of the unknown quantity to a single term.

IV. Divide both members of the resulting equation by the coefficient of the square of the unknown quantity.

The resulting equation, is called the **reduced equation.**

Solution of the Reduced Equations.

141. The solution of the reduced equation, consists in finding such values of the unknown quantity as will satisfy it, that is, when substituted for the unknown quantity, will make the two members equal. Every such value is called a **root.**

Two cases may arise: *first,* it may happen that $2p$, or the coefficient of the first power of the unknown quantity, is equal to 0; in this case, the equation is said to be **incomplete**: *secondly,* it may happen that the coefficient of the first power of the unknown quantity is not equal to 0; in this case the equation is said to be **complete.**

Incomplete equations, when reduced, have but two terms: one containing the square of the unknown quantity; the other, a known term.

Complete equations, when reduced, have three terms, viz.: a term containing the square of the unknown quantity, a term containing the first power of the unknown quantity, and a known term.

First Case. Incomplete Equations.

142. In this case, the reduced equation takes the form,

$$x^2 = q;$$

extracting the square root of both members, we have,

$$x = \pm \sqrt{q};$$

hence, we have the following rule for solving incomplete equations:

RULE.

Reduce the equation to the form, $x^2 = q$, *and extract the square root of both members.*

There will be two roots numerically equal, but having contrary signs. Denoting the first root by x', and the second by x'', we have

$$x' = +\sqrt{q}, \text{ and } x'' = -\sqrt{q}.$$

EXAMPLES.

Solve the following equations:

1. $x^2 + 5 = \frac{10x^2}{3} - 16.$ *Ans.* $x' = +3,\ x'' = -3.$

2. $3x^2 - 4 = 28 + x^2.$ *Ans.* $x' = +4,\ x'' = -4.$

3. $\frac{3x^2+5}{8} - \frac{x^2+29}{3} = 117 - 5x^2.$ *Ans.* $x' = +5,\ x'' = -5.$

4. $x^2 + ab = 5x^2.$ *Ans.* $x' = +\frac{1}{2}\sqrt{ab},\ x'' = -\frac{1}{2}\sqrt{ab}.$

5. $(x+a)^2 = 2ax + b.$ *Ans.* $x' = +\sqrt{b-a^2},\ x'' = -\sqrt{b-a^2}.$

6. $\frac{x+7}{x^2-7x} - \frac{x-7}{x^2+7x} = \frac{7}{x^2-73}.$ *Ans.* $x' = +9,\ x'' = -9.$

7. $x\sqrt{a+x^2} = b + x^2.$ *Ans.* $x' = +\frac{b}{\sqrt{a-2b}},\ x'' = \frac{-b}{\sqrt{a-2b}}.$

8. $\sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{x+2}{x-2}} = 4.$ *Ans.* $x' = +\frac{4}{3}\sqrt{3},\ x'' = -\frac{4}{3}\sqrt{3}.$

Second Case. Complete Equations.

143. The reduced form of the complete equation is,

$$x^2 + 2px = q;$$

adding p^2 to both members, (axiom 1°), we have,

$$x^2 + 2px + p^2 = q + p^2;$$

extracting the square root of both members, (axiom 5°), we have,

$$x + p = \pm\sqrt{q + p^2};$$

transposing p to the second member, we have,

$$x = -p \pm \sqrt{q + p^2};$$

hence, there are two roots, one corresponding to the *plus* sign of the radical, and the other to the *minus* sign; denoting these roots by x' and x'', we have,

$$x' = -p + \sqrt{q + p^2}, \text{ and } x'' = -p - \sqrt{q + p^2};$$

hence, we have the following rule for solving complete equations of the second degree:

RULE.

I. Reduce the equation to the form, $x^2 + 2px = q$, *by the rule.*

II. The first root is equal to half the coefficient of the second term, taken with a contrary sign, plus the square root of the second member increased by the square of half the coefficient of the second term.

III. The second root is equal to half the coefficient of the second term, taken with a contrary sign, minus the square root of the second member increased by the square of half the coefficient of the second term.

EXAMPLES.

1. Let it be required to solve the equation,

$$3x^2 - 14x + 15 = 0;$$

Reducing to the required form,

$$x^2 - \frac{14}{3}x = -5;$$

writing out the roots, we have,

$$x' = \frac{7}{3} + \sqrt{-5 + \frac{49}{9}}, \text{ and } x'' = \frac{7}{3} - \sqrt{-5 + \frac{49}{9}};$$

reducing, we have,

$$x' = \frac{7}{3} + \frac{2}{3} = 3, \text{ and } x'' = \frac{7}{3} - \frac{2}{3} = \frac{5}{3}.$$

These roots may be verified. Substituting 3 for x, in the given equation, we have,

$$3 \times (3)^2 - 14 \times 3 + 15 = 0;$$

substituting $\frac{5}{3}$ for x, in the same equation, we have

$$3 \times \left(\frac{5}{3}\right)^2 - 14 \times \frac{5}{3} + 15 = 0;$$

which shows that both 3, and $\frac{5}{3}$, are roots.

2. Again, let it be required to solve the equation,

$$4x - \frac{14 - x}{x + 1} = 14;$$

Clearing of fractions, and reducing to the required form,

$$x^2 - \frac{9}{4}x = 7;$$

writing out the roots, by the rule,

$$x' = \frac{9}{8} + \sqrt{7 + \frac{81}{64}}, \text{ and } x'' = \frac{9}{8} - \sqrt{7 + \frac{81}{64}};$$

reducing, we have,

$$x' = \frac{9}{8} + \frac{23}{8} = 4, \qquad \text{and } x'' = \frac{9}{8} - \frac{23}{8} = -\frac{7}{4};$$

which roots may be verified as before.

3. Given $\frac{3x + 4}{5} - \frac{30 - 2x}{x - 6} = \frac{7x - 14}{10}$, to find x.

Reducing and writing out the roots, we have,

$$x' = 24 + \sqrt{-432 + 576}, \quad \text{and } x'' = 24 - \sqrt{-432 + 576},$$

or, $\qquad x' = 36, \quad \text{and} \quad x'' = 12.$

In writing out the roots by the rule, it frequently happens that the quantity under the radical sign is made up of two fractions, or of an entire part and a fraction, as in examples 1 and 2. In such cases, the two parts must be reduced to the least common denominator that is a perfect square, and then added together.

Solve the following equations:

4. $5x^2 - 6x - 60 = 3.$ *Ans.* $x' = \frac{21}{5}, \; x'' = -3.$

5. $(x - 12)(x + 2) = 0.$

Ans. $x' = 12, \; x'' = -2.$

6. $ax^2 - bx = c.$

Ans. $x' = \frac{b + \sqrt{b^2 + 4ac}}{2a}, \quad x'' = \frac{b - \sqrt{b^2 + 4ac}}{2a}$

7. $\frac{10}{x} - \frac{14-2x}{x^2} = \frac{22}{9}$. *Ans.* $x' = 3,\ x'' = \frac{21}{11}$

8. $(x+2)^2 = 2x^2 + 8$. *Ans.* $x' = 2,\ x'' = 2$.

9. $4x^2 - 9x = 90$. *Ans.* $x' = 6,\ x'' = -\frac{15}{4}$.

10. $\frac{x-3}{x+5} - \frac{x+4}{x-7} = 2\frac{7}{9}$. *Ans.* $x' = -8\frac{21}{25},\ x'' = 4$.

11. $x^2 - (a+b)x + ab = 0$. *Ans.* $x' = a,\ x'' = b$.

12. $\frac{4x^2}{3} = \frac{x}{3} + 11$. *Ans.* $x' = 3,\ x'' = -\frac{11}{4}$.

13. $\frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{6}$. *Ans.* $x' = 2,\ x'' = -3$.

14. $\frac{x+4}{3} - \frac{4x+7}{9} = \frac{7-x}{x-3} - 1$.

Ans. $x' = 21,\ x'' = 5$.

15. $(x-1)^2 = 2(x^2+1)$.

Ans. $x' = -1,\ x'' = -1$.

16. $x^2\left(1 - \frac{1}{x}\right) = 8(x+2)$.

Ans. $x' = \frac{9+\sqrt{145}}{2},\ x'' = \frac{9-\sqrt{145}}{2}$.

17. $17x^2 + 19x - 1848 = 0$.

Ans. $x' = 9\frac{15}{17},\ x'' = -11$.

18. $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{3}{8} = 8$. *Ans.* $x' = 1\frac{1}{2},\ x'' = -\frac{5}{6}$.

19. $\frac{2x-10}{8-x} - \frac{x+3}{x-2} = 2$. *Ans.* $x' = 7,\ x'' = \frac{4}{5}$.

20. $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{35}$. *Ans.* $x' = 11,\ x'' = -13$.

21. $x + \frac{24}{x - 1} = 3x - 4.$ *Ans.* $x' = 5,\ x'' = -2.$

22. $\frac{x^2 + 1}{2x} + \frac{x - 1}{4} = 3x - 2.$

Ans. $x' = 1,\ x'' = -\frac{2}{9}.$

23. $x + \frac{1}{x} + 3\frac{(x - 1)}{4} = \frac{7}{x}.$

Ans. $x' = \frac{3 + \sqrt{681}}{14},\ x'' = \frac{3 - \sqrt{681}}{14}.$

24. $x^3 + (5 - x)^3 = 35.$ *Ans.* $x' = 3,\ x'' = 2.$

25. $\frac{1200}{x} = \frac{1200}{40 + x} + 5.$

Ans. $x' = 80,\ x'' = -120.$

Trinomial Equations.

144. Many equations of a higher degree than the second, may be reduced to the form of equations of the second degree, and then solved. One of the most important classes of such equations consists of what are called **trinomial equations.** Such equations contain three kinds of terms, viz.: terms involving the unknown quantity to any degree, terms involving the unknown quantity to a degree half as great, and known terms. Such, for example, as $x^6 - 4x^3 = 32$, $x^4 - 2x^2 = 3$.

Every trinomial equation may be reduced to the form of,

$$x^{2n} + 2px^n = q, \quad . \quad . \quad . \quad . \quad (1)$$

in the same way that equations of the second degree are reduced to the form of,

$$x^2 + 2px = q.$$

After an equation is reduced to the form (1) we may regard x^n as the unknown quantity, and then it may be solved by the rule given for the solution of equations of the second degree. Having found the values of x^n, we may find the values of x, by extracting the n^{th} root of these. To illustrate, let it be required to solve the equation,

26. $x^6 - 4x^3 = 32$.

This is of the required form. Writing out the values of x^3, by the rule, we have,

$$x'^3 = 2 + \sqrt{32 + 4} = 8, \text{ and } x''^3 = 2 - \sqrt{32 + 4} = -4.$$

Whence, by extracting the cube roots of these values, we have,

$$x' = \sqrt[3]{8} = 2, \text{ and } x'' = \sqrt[3]{-4}.$$

27. $x^4 - 2x^2 = 3$.

This is of the required form. Writing out the values of x^2, we have,

$$x'^2 = 1 + \sqrt{3 + 1} = 3, \text{ and } x''^2 = 1 - \sqrt{3 + 1} = -1.$$

Whence, by extracting the square roots of these roots, we have,

$$x' = \pm\sqrt{3}, \text{ and } x'' = \pm\sqrt{-1}.$$

28. $x^3 - x^{\frac{3}{2}} = 56$.

In this case $n = \frac{3}{2}$; hence, we have,

$$x'^{\frac{3}{2}} = \tfrac{1}{2} + \sqrt{56 + \tfrac{1}{4}} = 8, \text{ and } x''^{\frac{3}{2}} = \tfrac{1}{2} - \sqrt{56 + \tfrac{1}{4}} = -7;$$

squaring each value and extracting the cube root of the result, we have,

$$x' = 4, \text{ and } x'' = \sqrt[3]{49}.$$

Radical equations may sometimes be transformed so as to give rise to trinomial equations. Thus, let it be required to solve the equation,

29. $x + \sqrt{10x + 6} = 9.$

Multiplying both members by 10, and adding 6 to each member of the resulting equation, we have,

$$10x + 6 + 10\sqrt{10x + 6} = 96;$$

we may now regard $\sqrt{10x + 6}$ as the unknown quantity; solving with respect to this quantity, we have,

$$\sqrt{10x + 6} = -5 \pm \sqrt{121}$$

$$\therefore \quad \sqrt{10x + 6} = 6, \text{ and } -16;$$

consequently,

$$10x + 6 = 36, \text{ and } 256;$$

whence $x' = 3$, and $x'' = 25$.

Solve the following equations:

30. $x^4 - 8x^2 = 9.$
Ans. $x' = \pm 3, \quad x'' = \pm\sqrt{-1}.$

31. $x^6 + 20x^3 = 69.$ *Ans.* $x' = \sqrt[3]{3}, \quad x'' = \sqrt[3]{-23}.$

32. $\dfrac{\sqrt{4x + 20}}{4 + \sqrt{x}} = \dfrac{4 - \sqrt{x}}{\sqrt{x}}$
Ans. $x' = 4, \quad x'' = -\dfrac{64}{3}.$

33. $4x + 4\sqrt{x + 2} = 7.$ *Ans.* $x' = 4\frac{1}{4}, \quad x'' = \frac{1}{4}.$

34. $x \pm \sqrt{5x + 10} = 8.$ *Ans.* $x' = 18, \quad x'' = 3.$

–35. $ax + 2\sqrt{n^2x + nax^2} = (3x - 1)n.$
Ans. $x' = \dfrac{n}{n - a}, \quad x'' = \dfrac{n}{9n - a}.$

36. $x^4 - 74x^2 = -1225.$
Ans. $x' = \pm 7, \quad x'' = \pm 5.$

PROBLEMS.

1. Find two numbers whose difference is 8, and whose product is 128.

Let x denote the less number; then will $x + 8$ denote the greater number; from the conditions of the problem,

$$x(x + 8) = 128.$$

Solving this equation, we find,

$$x' = 8, \text{ and } x'' = -16; \text{ two solutions.}$$

Hence, the numbers are 8 and 16; also, -16 and -8. Either pair of numbers will satisfy the conditions of the problem, as may be seen by trial.

In using the second pair, we must remember that -16 is the less number, and due regard must be had to the signs.

2. A person traveled 105 miles at a uniform rate. On his return, he traveled 2 miles per hour slower, and was 6 hours longer in making the journey. How many miles did he travel per hour in the first instance?

Let x denote the number of miles he traveled per hour; then will $\frac{105}{x}$, denote the number of hours required to make the direct journey, and $\frac{105}{x - 2}$, the number required to make the return journey.

Since he traveled slower on the return journey, he will take a longer time; and from the conditions of the problem,

$$\frac{105}{x - 2} - \frac{105}{x} = 6;$$

whence, by solving the equation, we find,

$$x' = 7, \text{ and } x'' = -5.$$

The first solution is the answer to the given problem; the second solution is the answer to a problem which gives the same equation and which may be enunciated thus: A person traveled 105 miles, at a uniform rate. On his return, he traveled 2 miles per hour faster, and made the journey 6 hours sooner.

Denote the rate of travel on the outward journey, by $-x$: then the rate, on the return journey, will be denoted by $-x+2$; and, since he traveled slower on the outward journey,

$$-\frac{105}{x} - \frac{105}{-x+2} = 6; \text{ or } \frac{105}{x-2} - \frac{105}{x} = 6;$$

which is the same equation as that obtained before; hence, its solution should give proper answers to both problems.

When two answers are found to a problem, both of which do not satisfy the conditions of that problem, it will generally be found that *one* of them belongs to the problem as enunciated, and *the other* to a similar problem which gives rise to the same equation; hence, we are to take only that answer which satisfies the conditions of the given problem.

3. *A.* and *B.* set out from two points at the same time, and travel towards each other. On meeting, it appears that *A.* has traveled 30 miles more than *B.*, and that *A.* could travel *B.'s* distance in $4\frac{1}{6}$ days, and that *B.* could travel *A.'s* distance in 6 days. How far apart were they when they started?

Let x denote the number of miles that *B.* traveled; then will $x+30$ denote the number that *A.* traveled. Since *A.* can travel *B.'s* distance in $4\frac{1}{6}$ days, $\frac{x}{4\frac{1}{6}}$, or $\frac{6x}{25}$, will denote the number of miles that *A.* travels in 1 day. In like manner, $\frac{x+30}{6}$, will denote the number of miles that *B.* travels in 1 day. The whole distance that *A.* travels, divided by the distance he travels in one day, or $(x+30) \div \frac{6x}{25}$, gives the number of days that he travels. The whole distance that *B.* travels, divided by the distance he travels in one day, or, $x \div \frac{x+30}{6}$, gives the number of days that *B.* travels. But they travel the same number of days each. Hence,

$$\frac{x+30}{\left(\frac{6x}{25}\right)} = \frac{x}{\left(\frac{x+30}{6}\right)}.$$

Clearing of complex fractions,

$$\frac{(x+30)(x+30)}{6} = \frac{6x^2}{25}.$$

Whence, by solution, we find,

$$x' = 150, \text{ and } x'' = -\frac{150}{11}.$$

The first result only satisfies the conditions of the problem. Hence, *B.* travels 150 miles, *A.*, 180 miles, and both together, 330 miles, which is the answer to the given problem.

4. *A.*, *B.*, and *C.*, can together perform a piece of work, in a certain time. *A.* alone can perform the same work in 6 hours more, *B.* alone in 15 hours more, and *C.* alone in twice the time. In how many hours can they all perform it, working together?

Let x denote the number of hours required for all to perform it; then will $x+6$ denote the number of hours for *A.* to perform it; $x+15$ the number of hours for *B.* to perform it; and, $2x$ the number of hours for *C.* to perform it.

In 1 hour, *A.* can perform a portion of the work denoted by $\frac{1}{x+6}$; *B.* can perform a portion denoted by $\frac{1}{x+15}$; *C.* can perform a portion denoted by $\frac{1}{2x}$; and all together can perform a portion denoted by $\frac{1}{x}$; but as the sum of the parts that each can perform is equal to the part they all can perform, we have,

$$\frac{1}{x+6} + \frac{1}{x+15} + \frac{1}{2x} = \frac{1}{x},$$

which is the equation of the problem.

Solving, we have,

$$x' = 3 \quad \text{and} \quad x'' = -10.$$

The first value is the one that satisfies the conditions of the problem; hence, the answer is 3 *hours*.

5. There are two numbers whose sum is 40, and the sum of their squares is 818. What are the numbers? *Ans.* 17, and 23.

6. The difference of two numbers is 9; and their sum, multiplied by the greater, is equal to 266. What are the numbers? *Ans.* 14, and 5.

7. The sum of two numbers is 73, and their product 732. What are the numbers? *Ans.* 61, and 12.

8. The sum of two numbers is a, and their product is b. What are the numbers?

$$\textit{Ans.}\ \frac{a}{2} + \sqrt{-b + \frac{a^2}{4}}, \text{ and } \frac{a}{2} - \sqrt{-b + \frac{a^2}{4}}.$$

9. A person travels 48 miles at a uniform rate; a second person travels the same distance two hours sooner, and travels 2 miles per hour more than the first. At what rate does the first travel? *Ans.* 6 miles.

10. *A.* and *B.* start at the same time to travel 150 miles. *A.* travels 3 miles an hour faster than *B.*, and finishes his journey $8\frac{1}{3}$ hours before him. How many miles does *A.* travel per hour? *Ans.* 9 miles.

11. A hollow cubical box, whose sides are three inches thick, requires for its construction $27\frac{1}{8}$ cubic feet of material. How many cubic feet of water will it contain? *Ans.* 64 cubic feet.

12. Two square courts are paved with stones a foot square; the larger court is 12 feet longer than the smaller one, and the number of stones in both pavements is 2120. How long is the smaller pavement?

Ans. 26 feet.

13. A person distributes 120 dollars amongst a certain number of people. The next day he distributes the same sum amongst a number of people greater by 2. Each of the latter receives 2 dollars less than each of the former. How many were there in each case?

Ans. 10, and 12.

14. *A.* and *B.* set out to meet each other, being 320 miles apart. *A.* traveled 8 miles a day more than *B.*, and the number of days before they met, was equal to half the number of miles that *B.* traveled in 1 day. How many miles did each travel per day?

Ans. *A.*, 24 miles; *B.*, 16 miles.

15. A passenger and freight train set out at the same time, the former from New York, and the latter from Albany, distant from each other 144 miles. The passenger train arrived in Albany two hours after they met, and the freight train arrived in New York 8 hours after they met. At what rate did each run?

Ans. 24, and 12 miles.

16. A regiment was ordered to furnish 216 men for duty, by detailing the same number of men from each company. But three companies having been detached, the remaining ones had to furnish each 12 men more to make up the required number. How many companies were there in the regiment? *Ans.* 9.

17. Two partners, *A.* and *B.*, gained 360 dollars. *A.'s* money was in trade 12 months, and he received, for principal and profit, 520 dollars. *B.'s* money was 600 dollars, and was in trade 16 months. How much capital had *A.*? *Ans.* 400 dollars.

18. *A.* and *B.* travel, at the same rate, towards New York. At the 50th mile-stone from New York, *A. overtakes* a flock of geese, traveling $1\frac{1}{2}$ miles an hour, and 2 hours afterwards meets a stage-coach, traveling $2\frac{1}{4}$ miles per hour. *B. overtakes* the geese at the 45th mile-stone, and *meets* the coach 40 minutes before reaching the 31st mile-stone. What is the distance between *A.* and *B.*?

SOLUTION.

Let x denote the rate of *A.'s* and *B.'s* travel, and suppose the circumstances of the problem to commence when *A.* is at the 50th mile-stone.

When *B.* overtakes the geese, he will have traveled $3\frac{1}{3}x$ miles, 5 of which coincide with 5 of *A.'s* miles. Hence, the distance between *A.* and *B.*, is $\left(\frac{10x}{3} - 5\right)$ miles.

A. meets the coach at the $(50 - 2x)$ mile-stone, and *B* meets it at the $\left(31 + \frac{2x}{3}\right)$ mile-stone; the coach travels, meantime, $\left(\frac{8}{3}x - 19\right)$ miles. Hence, dividing by the rate of the coach, we have, $\frac{32x - 228}{27}$, the number of hours between the meetings. Adding to $\frac{8}{3}x - 19$, the distance that *A.* travels in that time, we have, $\left(\frac{8x - 57}{3} + \frac{32x^2 - 228x}{27}\right)$ miles, for the distance between *A.* and *B.* Hence, from the conditions of the problem,

$$\frac{8x - 57}{3} + \frac{32x^2 - 228x}{27} = \frac{10x - 15}{3};$$

$$\therefore\ x = 9, \text{ and } \frac{10x - 15}{3} = 25, \text{ the number of miles required.}$$

General Properties of Equations of the Second Degree.

145. It has been shown, (Art. 140), that every equation of the second degree may be reduced to the form,

$$x^2 + 2px = q \quad . \quad . \quad . \quad . \quad (1)$$

adding p^2 to both members of equation (1), and then transposing all the terms of the *second* member to the first, we have,

$$(x^2 + 2px + p^2) - (q + p^2) = 0 \quad . \quad . \quad (2)$$

Factoring the first member, according to the principle that the difference of the squares of two quantities is equal to the sum of the quantities multiplied by their difference, we have,

$$(x + p + \sqrt{q + p^2}) \times (x + p - \sqrt{q + p^2}) = 0 \; . \; (3)$$

Equation (3) may be satisfied in two ways, and only in two ways, viz.: we may make the second factor of the first member equal to 0, and we may make the first factor equal to 0.

Placing the second factor equal to 0, we have,

$$x + p - \sqrt{q + p^2} = 0 ; \quad \therefore \; x = -p + \sqrt{q + p^2} \; . \; (4)$$

Placing the second factor equal to 0, we have,

$$x + p + \sqrt{q + p^2} = 0 ; \quad \therefore \; x = -p - \sqrt{q + p^2} \; . \; (5)$$

These suppositions give the two roots already found, and those only; hence, we deduce this principle:

1°. *Every equation of the second degree has two roots, and only two.*

If we examine Equation (3), we see that its first member is composed of two factors of the first degree with respect to x, the first term of each factor being the unknown quantity, and the second terms being the two roots, each taken with a contrary sign; hence, we deduce the following principle:

2°. *If all the terms are transposed to the first member, that member may be resolved into two factors of the first degree with respect to the unknown quantity, the first term of each factor being the unknown quantity, and the second terms being the two roots, each taken with its sign changed.*

The reverse of this principle, enables us to form an equation, when its roots are given, for which operation we may write the following

RULE.

Subtract each root from the unknown quantity, multiply the results together, and place the product equal to 0.

EXAMPLES.

1. Required the equation whose roots are 2 and -3.

By the rule, $(x-2)(x+3) = 0$, whence,

$$x^2 + x - 6 = 0; \text{ or, } x^2 + x = 6.$$

2. Required the equation whose roots are 5 and $-\frac{2}{3}$.

Ans. $x^2 - \dfrac{13}{3}x = \dfrac{10}{3}$.

3. Required the equation whose roots are a and b.

Ans. $x^2 - (a+b)x = -ab$.

4. Required the equation whose roots are $\frac{3}{7}$ and $\frac{7}{3}$.

Ans. $x^2 - \frac{58}{21}x = -1.$

5. Required the equation whose roots are -7 and -3.

Ans. $x^2 + 10x = -21.$

6. Required the equation whose roots are $-\frac{3}{4}$ and $\frac{5}{2}$.

Ans. $x^2 - \frac{7}{4}x = \frac{15}{8}.$

Let us resume the consideration of equation (1), and its roots:

Adding the two roots together, we have,

$$(-p + \sqrt{q+p^2}) + (-p - \sqrt{q+p^2}) = -2p;$$

multiplying the two roots together, we have,

$$(-p + \sqrt{q+p^2}) \times (-p - \sqrt{q+p^2}) = -q.$$

From these results, we deduce the following additional principles:

3°. *The algebraic sum of the two roots is equal to the coefficient of the second term, with its sign changed.*

4°. *The product of the two roots is equal to the second member, with its sign changed.*

If q is negative and numerically greater than p^2, the quantity under the radical sign, in equations (4) and (5), will be negative. In this case both roots are imaginary, (Art. 118); hence, the following principle:

5°. *If the second member is negative and numerically* *greater than the square of half the coefficient of the second term, both of the roots will be imaginary.*

These principles are used in discussing the general equation of the second degree.

Discussion of the General Equation.

146. To discuss the general equation,

$$x^2 + 2px = q,$$

we make every possible hypothesis, and interpret the corresponding results. It is plain, *first*, that both p and q may be positive; *second*, that p may be negative, and q positive; *third*, that p may be positive, and q negative; and *fourth*, that p and q may be negative; these hypotheses give,

The Four Forms.

$$x^2 + 2px = q \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x^2 - 2px = q \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$x^2 + 2px = -q \quad . \quad . \quad . \quad . \quad (3)$$

$$x^2 - 2px = -q \quad . \quad . \quad . \quad . \quad (4)$$

First Form.

Since q is positive, the product of the roots must be negative, (principle 4°); hence, *the roots have contrary signs;* again, since $2p$ is positive, the algebraic sum of the roots is negative, (principle 3°); hence, *the negative root is numerically the greater.*

Second Form.

For the same reason as before, *the roots have contrary signs;* since $2p$ is negative, the algebraic sum of the roots is positive, (principle 3°); hence, *the positive root is numerically the greater.*

Third Form.

Here, q is negative, and consequently the product of the roots is positive, (principle 4°); hence, *the roots have the same sign;* and, since $2p$ is positive, the sum of the roots must be negative, (principle 3°); hence, *both roots are negative.*

Fourth Form.

For the same reason as before, *the roots have the same sign;* because $2p$ is negative, the sum of the roots is positive, (principle 3°); hence, *the roots are both positive.*

If we suppose $p = 0$, and q not equal to 0, the first and second forms reduce to

$$x^2 = q.$$

In this case, the roots have contrary signs, and their sum is equal to 0; hence, *they are equal, with contrary signs.*

Under the same supposition, the third and fourth forms become,

$$x^2 = -q.$$

In this case, the roots have the same sign, and their sum is 0, which is impossible. But, in this case, the roots are imaginary, (principle 5°); hence, imaginary roots indicate that the supposition which gives rise to them is impossible, or absurd.

If we suppose $q = 0$, and p not equal to 0, the first and third forms reduce to

$$x^2 + 2px = 0,$$

and the second and fourth forms reduce to

$$x^2 - 2px = 0.$$

In both cases, the product of the roots is equal to 0, which shows that one of them must be 0. In the first case, the second root is equal to $-2p$, and in the second, it is equal to $+2p$.

If we suppose, $p = 0$, and $q = 0$, all the forms reduce to

$$x^2 = 0.$$

In this case, the product of the roots is 0, hence, one of them must be 0, and their sum is equal to 0; hence, the other is also equal to 0.

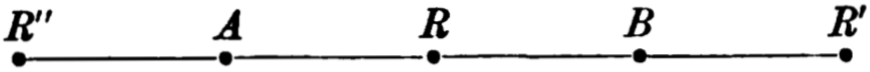
Solution and Discussion of the Problem of the Lights.

147. The solution of the problem of the lights depends on the following principle of physics:

Principle.—The intensity of a light at any distance is equal to its intensity at the distance 1, *divided by the square of the distance.*

Problem. Having given the intensities of two lights, at the distance 1, and the distance between the lights, it is required to find a point on the line joining them which is equally illuminated by both.

SOLUTION. Let the first light be at A, the second light at B, and let $R''R'$ be a straight line passing through A and B. Assume A as the origin of distances, and call all distances to the right *positive*, then will all distances to the left be *negative*.



Denote the distance between the lights by c, the intensity of the light A, at the distance 1, by a, and that of the light B, at the distance 1, by b. Suppose the point R, to be equally illuminated, and denote its distance from the origin A, by x; then will the distance BR, be equal to $c - x$.

Since the intensity of the light A, at the distance 1, is a, at the distance x it will be $\frac{a}{x^2}$, in accordance with the physical principle enunciated; and since the intensity of the light B, at the distance 1, is b, at the distance $c - x$ it will be equal to $\frac{b}{(c-x)^2}$. From the conditions of the problem, these two expressions must be equal; hence,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}; \quad \text{or,} \quad \frac{(c-x)^2}{x^2} = \frac{b}{a};$$

extracting the square root of both members, we have,

$$\frac{c-x}{x} = \pm\frac{\sqrt{b}}{\sqrt{a}};$$

taking the upper sign, we have, for the first value of x,

$$x = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) \quad . \quad . \quad (1)$$

taking the lower sign, we have, for the second value of x,

$$x = \frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}} = c\left(\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}\right) \quad . \quad . \quad (2)$$

From the nature of the problem, both a and b are positive, and consequently these values of x are always real; hence, we see that there are two points on the line AB that are equally illuminated by the lights.

Discussion. The quantities a, b and c are arbitrary, but inasmuch as the conditions of the problem involve the necessity of two lights, none of these quantities can be 0. From the fact that the left hand light was taken as the origin of distances, the value of c must always be positive. Hence, there are three, and only three, suppositions that can be made on a, b, and c:

1°. $c > 0$, and $a > b$;
2°. $c > 0$, and $a < b$; and
3°. $c > 0$, and $a = b$.

First Supposition. $c > 0$, *and* $a > b$.

In this case, $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, is a proper fraction, and therefore less than 1; and because the denominator is less than twice the numerator, it is greater than $\frac{1}{2}$; hence, the first value of x is less than c, and greater than $\frac{1}{2}c$; which shows that the first point of equal illumination is between the two lights, and nearer the feebler one.

The denominator of the fraction $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is less than the numerator, and consequently the fraction itself is greater than 1, that is, the second value of x is greater than c; which shows that the second point of equal illumination is in the prolongation of AB, and on the side of the feebler light.

Second Supposition. $c > 0$, *and* $a < b$.

In this case the fractional factor of the first value of x is positive and less than $\frac{1}{2}$; hence, the first value of x is positive and less than $\frac{1}{2}c$; which shows that the first point of equal illumination is between the two lights, and nearer the feebler one.

The fractional factor of the second value of x is negative, and consequently the second value of x is negative; which shows that the second point of equal illumination is to the left of A, that is, it is on the prolongation of BA, and on the side of the feebler light.

The results of the first and second suppositions agree with each other. In both cases the *first* point is between the two lights, and nearer the feebler one, and the *second* point is in the prolongation of the line joining the lights and on the side of the feebler one.

Third Supposition. $c > 0$, *and* $a = b$.

In this case the fractional factor of the first value of x is equal to $\frac{1}{2}$, and consequently the first value of x is equal to $\frac{1}{2}c$; which shows that the first point of equal illumination is midway between the two lights.

The fractional factor of the second value of x is equal to $\frac{\sqrt{a}}{0}$ or to ∞; which shows that the second point of equal illumination is at a distance from A greater than any assignable distance, that is, there is but one point on the line of the lights that is equally illuminated by them.

The preceding results have been interpreted in accordance with the principles already explained. It is obvious that these interpretations are in strict agreement with the conditions implied in the several suppositions.

II. EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

Explanation.

148. Two equations of the second degree containing two unknown quantities cannot, in general, be solved by preceding methods. When one is of the second and the other of the first degree, they may be solved. There are some special cases in which the solution may be effected, even when both equations are of the second degree.

First Special Case.

149. Having given one equation of the second degree and one of the first degree, each containing two unknown quantities, we may proceed as follows:

EXAMPLES.

1. Take the two equations,

$$x^2 + 12xy + y^2 = 85 \quad . \quad . \quad . \quad (1)$$

$$x + 3y = 11 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Find the value of x in terms of y from (2), and substituting it in (1), we have,

$$121 - 66y + 9y^2 + 132y - 36y^2 + y^2 = 85;$$

or, by reduction, $$y^2 - \frac{33}{13}y = \frac{18}{13}:$$

whence, $$y' = 3, \text{ and } y'' = -\tfrac{6}{13},$$

which, substituted in (2), gives,

$$x' = 2, \text{ and } x'' = 12\tfrac{5}{13}.$$

In like manner, other similar groups of equations may be solved.

Solve the following groups of simultaneous equations:

2. $\left.\begin{array}{l} x^2 + y^2 = 202 \\ x + y = 20 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x' = 11,\ x'' = 9. \\ y' = 9,\ y'' = 11. \end{array}\right.$

3. $\left.\begin{array}{l} x^2 + y^2 = 394 \\ x - y = 2 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x' = 15,\ x'' = -13. \\ y' = 13,\ y'' = -15. \end{array}\right.$

4. $\left.\begin{array}{l} x^2 - 2xy + y^2 = 9 \\ x + y = 11 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x' = 7,\ x'' = 4. \\ y' = 4,\ y'' = 7. \end{array}\right.$

5. $\left.\begin{array}{l} x + y = 6 \\ x^2 + y^2 = 26 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x' = 5,\ x'' = 1. \\ y' = 1,\ y'' = 5. \end{array}\right.$

6. $\left.\begin{array}{l} x^2 - y^2 = 16 \\ x + y = 8 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x' = 5,\ x'' = 5. \\ y' = 3,\ y'' = 3. \end{array}\right.$

Second Special Case.

150. Having given two equations of the second degree which are homogeneous with respect to the unknown quantities, we may proceed as follows:

Take the equations,

7. $$x^2 + xy = 10 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$y^2 + xy = 15 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Assume $y = nx$, n being an auxiliary unknown quantity. Substituting in (1) and (2), we have,

$$x^2 + nx^2 = 10. \qquad \therefore \ x^2 = \frac{10}{n+1} \quad . \quad . \quad . \quad (3)$$

$$n^2x^2 + nx^2 = 15. \qquad \therefore \ x^2 = \frac{15}{n(n+1)}; \quad . \quad . \quad (4)$$

equating these values of x^2, we find,

$$\frac{10}{n+1} = \frac{15}{n(n+1)}; \qquad \therefore \ n = \frac{3}{2};$$

substituting this value of n in (3), and the resulting value of x in (1), we have,

$$x = 2, \ \text{and} \ y = 3.$$

Only a single pair of values of x and y are deduced. The complete solution would give four pairs of values.

In the same way, similar groups of equations may be solved.

Solve the following groups of equations:

8. $$\left.\begin{aligned} x^2 + y^2 &= 61 \\ x^2 - xy &= 6 \end{aligned}\right\} \qquad \textit{Ans.} \ \begin{cases} x = 6. \\ y = 5. \end{cases}$$

9. $$\left.\begin{aligned} x^2 + xy + y^2 &= 37 \\ x^2 - xy + y^2 &= 13 \end{aligned}\right\} \qquad \textit{Ans.} \ \begin{cases} x = 3. \\ y = 4. \end{cases}$$

10. $$\left.\begin{aligned} x^2 - 2xy &= 5 \\ y^2 + x^2 &= 29 \end{aligned}\right\} \qquad \textit{Ans.} \ \begin{cases} x = 5. \\ y = 2. \end{cases}$$

11. $$\left.\begin{aligned} 3x^2 &= 2xy + 24 \\ y^2 &= xy - 3 \end{aligned}\right\} \qquad \textit{Ans.} \ \begin{cases} x = 4. \\ y = 3. \end{cases}$$

12. $$\left.\begin{aligned} 4xy - 3y^2 &= 64 \\ 2xy + 2x^2 - y^2 &= 138 \end{aligned}\right\} \qquad \textit{Ans.} \ \begin{cases} x = 7. \\ y = 4. \end{cases}$$

Miscellaneous Cases.

151. Many other equations of the second degree may be so transformed as to come within the rules for solution. Equations of a higher degree than the second may, also, in certain cases, be reduced, by transformation, to such forms as to come within the rules already given. No general principles can be laid down for making these transformations, each case requiring to be treated in a manner peculiar to itself. A few examples are given, to illustrate some of the methods of solution.

EXAMPLES.

13. $$x^2 + y^2 + x + y = 922 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\sqrt{xy} = 20 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Squaring (2), and multiplying by 2,

$$2xy = 800\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

adding (1) and (3),

$$(x^2 + 2xy + y^2) + (x + y) = 1722\,;$$

regarding $(x + y)$ as a single quantity, we have, by the rule,

$$x + y = -\tfrac{1}{2} \pm \sqrt{1722 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \tfrac{83}{2}\,;$$
$$\therefore \quad x + y = 41, \quad \text{and} \quad x + y = -42\,;$$

taking the first value of $x + y$, and combining with (3), we have,

$$x = 25, \quad y = 16.$$

14. $$\left.\begin{array}{ll} x + y + \sqrt{x + y} = 12 \quad . \quad . \quad . \quad . \quad . & (1) \\ x^3 + y^3 \qquad\qquad = 189 \quad . \quad . \quad . \quad . \quad . & (2) \end{array}\right\}$$

From (1) taking only the first value, we have,

$$\sqrt{x + y} = -\tfrac{1}{2} + \sqrt{12 + \tfrac{1}{4}} = -\tfrac{1}{2} + \tfrac{7}{2} = 3:$$
$$\therefore \quad x + y = 9\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

dividing (2) by (3), member by member, we have,

$$x^2 - xy + y^2 = 21 ; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

squaring (3), we have,

$$x^2 + 2xy + y^2 = 81 ; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

subtracting (4) from (5), we have,

$$3xy = 60 ; \quad \text{or,} \quad xy = 20 ; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

combining (6) and (3), we have,

$$x = 5, \quad \text{and} \quad y = 4.$$

15. $x - y = 2$ . . . . (1)

$x^4 + y^4 = 272$ . . . . (2)

Raising both members of (1) to the 4th power, we have,

$$x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = 16 \quad . \quad . \quad . \quad (3)$$

subtracting (3) from (2), and factoring,

$$xy(4x^2 - 6xy + 4y^2) = 256 ; \quad . \quad . \quad (4)$$

multiplying (1) by 2, squaring, multiplying by xy, and subtracting from (4), we have,

$$2x^2y^2 = 256 - 16xy ;$$

or,

$$x^2y^2 + 8xy = 128.$$

$$\therefore \; xy = -4 \pm \sqrt{128 + 16} = -4 \pm 12 ;$$

or,

$$xy = 8, \quad xy = -16.$$

Taking the first value $xy = 8$, and combining with (1), we find,

$$x = 4, \quad y = 2.$$

16. $\left.\begin{aligned} x^2 + 2xy + y + 3x &= 73 \\ y^2 + x + 3y &= 44 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 4. \\ y &= 5. \end{aligned}\right.$

17. $\left.\begin{aligned} xy &= 6 \\ 3x^2 - 7y^2 + 1 &= 0 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x &= 3. \\ y &= 2. \end{aligned}\right.$

18. $$\left.\begin{aligned} x^4 - 2x^2y + y^2 &= 49 \\ x^4 - 2x^2y^2 + y^4 - x^2 + y^2 &= 20 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 3. \\ y &= 2. \end{aligned}\right.$$

19. $$\left.\begin{aligned} \frac{x}{y} - \frac{y}{x} &= \frac{11}{30} \\ x^2 + xy &= 66 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 6. \\ y &= 5. \end{aligned}\right.$$

20. $$\left.\begin{aligned} x^2y^4 + y^2 &= 10 \\ xy^2 + y &= 4 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 3. \\ y &= 1. \end{aligned}\right.$$

21. $$\left.\begin{aligned} x^3 + y^3 &= 189 \\ x^2y + xy^2 &= 180 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 5. \\ y &= 4. \end{aligned}\right.$$

22. $$\left.\begin{aligned} \frac{x+y}{x-y} &= a^2 \\ x^2 - y^2 &= b^2 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= \frac{b}{2a}(a^2+1). \\ y &= \frac{b}{2a}(a^2-1). \end{aligned}\right.$$

23. $$\left.\begin{aligned} 9x^2 &= 4y^2 \\ 3xy + 2x + y &= 485 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 10. \\ y &= 15. \end{aligned}\right.$$

24. $$\left.\begin{aligned} x^2 + y^2 - x - y &= 78 \\ xy + x + y &= 39 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 9. \\ y &= 3. \end{aligned}\right.$$

25. $$\left.\begin{aligned} \frac{1}{y} - \frac{1}{x} &= \frac{1}{4} \\ x^2y - xy^2 &= 16 \end{aligned}\right\} \quad \textit{Ans.} \left\{\begin{aligned} x &= 4. \\ y &= 2. \end{aligned}\right.$$

PROBLEMS.

1. Find two numbers, such that their product, added to their sum, shall be 47, and their sum, taken from the sum of their squares, shall leave 62.

Let x and y denote the numbers; then, from the conditions of the problem,

$$(x + y) + xy = 47 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x^2 + y^2 - (x + y) = 62 ; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

multiplying equation (1) by 2, we have,

$$2xy + 2(x + y) = 94 ; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

adding (2) and (3), we have,

$$x^2 + 2xy + y^2 + (x + y) = 156 ;$$

or,

$$(x + y)^2 + (x + y) = 156 ; \quad . \quad . \quad . \quad . \quad (4)$$

solving (4), with respect to $x + y$, and taking the first value of $x + y$, we have,

$$x + y = -\tfrac{1}{2} + \sqrt{156 + \tfrac{1}{4}} = 12 ; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

substituting in (1), we have,

$$xy = 47 - 12 = 35 ; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

combining (5) and (6),

$x = 5$, and $y = 7$: the numbers required.

2. The sum of two numbers is 7, and the sum of their cubes is 91. What are the numbers?

Ans. 3 and 4.

3. Required two numbers, whose product is equal to the square of two thirds the first, and the difference of whose squares is greater, by 1, than the square of twice the second. *Ans.* 9 and 4.

4. Find two numbers, whose sum, multiplied by the greater, is 209, and whose difference, multiplied by the less, is 24. *Ans.* 11 and 8.

5. Find two numbers, such that the sum of their squares is equal to 181, and the difference of their squares equal to 19. *Ans.* 9 and 10.

III. INEQUALITIES.

Definitions and Explanations.

152. An **inequality** is an algebraic expression indicating that one quantity is greater or less than another. Thus, $a > b + c$, and $a < b - c$ are inequalities; the former indicates that a is greater than $b + c$, and the latter that a is less than $b - c$. The two parts connected by the sign of inequality are called **members**; that on the left of the sign is called the *first* member, and that on the right of the sign is called the *second* member.

Of two unequal quantities, that is algebraically the **greater** which is nearer to $+\infty$, thus

$$3 > 2, \quad \text{and} \quad -2 > -3.$$

Two inequalities subsist in the **same sense** when the greater quantity is either in the first member of both, or in the second member of both; they subsist in a **contrary sense** when the greater quantity is in the first member of one and in the second member of the other. Thus, the inequalities,

$$5 > 3 \quad \text{and} \quad 7 > 2,$$

subsist in the same sense; whilst the inequalities,

$$5 < 7 \quad \text{and} \quad 3 > 1$$

subsist in a contrary sense.

Transformations.

153. The following transformations of inequalities follow from the preceding definitions and explanations:

1°. *If the same quantity is added to, or subtracted from, both members of an inequality, the sense of the inequality will not be changed.*

Thus, if we have the inequality $13 > 12$, we also have the inequalities

$$13 + 2 > 12 + 2, \quad \text{and} \quad 13 - 3 > 12 - 3.$$

Hence, *we may transpose a term from one member to the other by changing its sign.*

2°. *If both members of an inequality are either multiplied or divided by a positive quantity, the sense of the inequality will not be changed.*

Thus, if we have the inequality $12 > 6$, we also have the inequalities

$$12 \times 3 > 6 \times 3 \quad \text{and} \quad \frac{12}{4} > \frac{6}{4}.$$

This principle enables us to clear an inequality of fractions by the rule for clearing an equation of fractions.

3°. *If we change the signs of both members of an inequality, we must change the sense of the inequality.*

Thus, if $3 > 2$, we have, $-3 < -2$.

Solution of Inequalities.

154. The **solution** of an inequality is the operation of finding an inequality in which the unknown quantity shall form one member; the other member is then a **limiting value** of that quantity.

The method of solution is indicated below:

Let it be required to find a limiting value of x from the inequality

$$-\frac{x}{3} + 4 < \frac{x}{4} - 3:$$

Multiplying both members by 12 (principle 2°), we have,

$$-4x + 48 < 3x - 36;$$

transposing and reducing (principle 1°), we have,

$$-7x < -84;$$

changing the signs of both members (principle 3°), we have,

$$7x > 84;$$

dividing both members by 7 (principle 2°), we have,

$$x > 12.$$

EXAMPLES.

Find limiting values of x from the following inequalities:

1. $\frac{3}{4}x - 7 > \frac{1}{5}x - 5.$ *Ans.* $x > 3\frac{7}{11}.$

2. $\frac{1}{4}x + 14 > \frac{3}{2}x + 1.$ *Ans.* $x < 10\frac{2}{5}.$

3. $3x - 12 + \frac{x}{4} > -2x + 9.$ *Ans.* $x > 4.$

CHAPTER X.

RATIO, PROPORTION, AND SERIES.

I. RATIO, AND PROPORTION.

Explanation.

155. We are said to **measure** a quantity when we find how many times it contains a quantity of the same kind, taken as a standard; the latter quantity is called the **unit of measure.**

As the unit is assumed to be a quantity whose value is known before the measure is made, we call it the **antecedent**; because the value of the quantity to be measured is found in terms of this antecedent, we call it a **consequent.**

Mathematically speaking, the measurement is performed by dividing the consequent by the antecedent; the result is an abstract number which we call a **ratio.** This ratio, prefixed to the unit employed, is the expression for the value of the quantity to be measured; hence, we have the following

Definition.

156. The **ratio** of one quantity to another is the result obtained by dividing the second quantity by the first. Thus, the ratio of a to b is equal to $\frac{b}{a}$, in which b is the quantity to be measured or the *consequent*, and a is the unit or *antecedent*.

Different Methods of Expressing a Ratio.

157. The ratio of a to b may be expressed by the symbol $\frac{b}{a}$, or, it may be written $a:b$; in the latter case the sign :, stands for *is contained in*. In both cases a is the *unit*, or *antecedent*, and b is the *consequent*. The antecedent and consequent are called **terms** of the ratio, the antecedent being the *first* term and the consequent being the *second* term.

Definitions.

158. A **proportion** is an expression of equality between two ratios.

A proportion may be written in two ways. Thus, if the ratio of a to b is equal to the ratio of c to d, we may indicate this equality by either of the following expressions:

$$\frac{b}{a} = \frac{d}{c}, \quad \text{or,} \quad a : b :: c : d.$$

Either of these expressions indicates that the ratio of a to b is equal to the ratio of c to d. The former may be read a is contained in b as many times as c is contained in d; the latter may be read a is to b, as c is to d. We may reverse these readings without error.

159. There are four *terms* in every proportion which have received different names with respect to each other. The first and third are **antecedents**; the second and fourth are **consequents.** The first and fourth are **extremes**; the second and third are **means.** The first and second form the **first couplet**; the third and fourth form the **second couplet.** The fourth term, is called a **fourth proportional to the other three.**

When the second term is equal to the third, it is said to be a **mean proportional** between the other two. In this case, there are but three different terms in the proportion, and the last term is said to be a **third proportional to the other two.**

In the proportion,

$$\frac{b}{a} = \frac{d}{c}, \quad \text{or} \quad a : b :: c : d,$$

a and c are *antecedents,* b and d *consequents,* a and d *extremes,* b and c *means;* $\frac{b}{a}$, or $a : b$, is the *first couplet,* $\frac{d}{c}$, or $c : d$, is the *second couplet,* and d is a *fourth proportional* to a, b, and c. Also, in the proportion,

$$\frac{b}{a} = \frac{c}{b}, \quad \text{or} \quad a : b :: b : c,$$

b is a *mean proportional* between a and c, and c is a *third proportional* to a and b.

160. Quantities are in proportion, **by alternation,** when antecedent is compared with antecedent, and consequent with consequent.

161. Quantities are in proportion, **by inversion,** when antecedents are made consequents, and consequents are made antecedents.

162. Quantities are in proportion, **by composition,** when the sum of antecedent and consequent is compared with either antecedent or consequent.

163. Quantities are in proportion, **by division,** when the difference of antecedent and consequent is compared with either antecedent or consequent.

164. Two varying quantities are **reciprocally,** or **inversely** proportional, when one is increased as many times as the other is diminished. In this case, their product is a fixed quantity, as $xy = m$.

165. Equimultiples of two quantities, are the results obtained by multiplying both by the same quantity. Thus, ma, and mb, are *equimultiples* of a and b, whatever may be the value of m.

Principles of Proportion.

166. We shall demonstrate some of the most important principles of proportions, adopting both methods of writing proportions.

Assume the proportion,

$$a : b :: c : d; \quad \text{or,} \ \frac{b}{a} = \frac{d}{c}; \quad . \quad . \quad . \quad (1)$$

clearing of fractions, we have,

$$bc = ad; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

hence, the following principles:

1°. *If four quantities are in proportion, the product of the means is equal to the product of the extremes.*

Conversely, if we divide both members of (2) by ca, we have, $\frac{b}{a} = \frac{d}{c}$; or, $a : b :: c : d$; hence,

If the product of two quantities is equal to the product of two other quantities, the first two may be made the means, and the second two the extremes, of a proportion.

It follows, that of three proportional quantities, *the square of the mean is equal to the product of the extremes.*

If we multiply both members of (1) by $\frac{c}{b}$, and reduce the result to its simplest form, we have,

$$\frac{c}{a} = \frac{d}{b}; \text{ or, } a : c :: b : d \quad . \quad . \quad (3)$$

whence, the following principle:

2°. *If four quantities are in proportion, they will be in proportion by alternation.*

Let us assume the proportion,

$$a : b :: g : f; \text{ or, } \frac{b}{a} = \frac{f}{g} \quad . \quad . \quad (4)$$

Comparing (4) with (1), we see that their first members are equal; hence, their second members must also be equal; that is,

$$\frac{d}{c} = \frac{f}{g}; \text{ or, } c : d :: g : f; \quad . \quad . \quad . \quad (5)$$

hence, the following principle:

3°. *If the first couplets of two proportions are the same, the second couplets will form a proportion.*

Consequently, by alternation, *if the antecedents of two proportions are the same in both, the consequents will be in proportion.*

If we take the reciprocals of both members of (1), we have,

$$\frac{a}{b} = \frac{c}{d}; \text{ or, } b : a :: d : c; \quad . \quad . \quad . \quad (6)$$

hence, the following principle:

4°. *If four quantities are in proportion, they will be in proportion by inversion.*

If we add 1 to both members of (1), and also subtract 1 from both members, we shall have,

$$\frac{b}{a} + 1 = \frac{d}{c} + 1, \text{ and } \frac{b}{a} - 1 = \frac{d}{c} - 1;$$

whence, by reducing to a common denominator, we have,

$$\left.\begin{array}{l} \dfrac{b+a}{a} = \dfrac{d+c}{c}, \quad \text{and} \quad \dfrac{b-a}{a} = \dfrac{d-c}{c} \\ \text{or } a : b + a :: c : d + c, \text{ and } a : b - a :: c : d - c \end{array}\right\}; (7)$$

hence, the following principle:

5°. *If four quantities are in proportion, they will be in proportion by composition or by division.*

If we multiply both terms of the ratio $\frac{b}{a}$, by m, its value will not be changed (Art. 61), and we have,

$$\frac{mb}{ma} = \frac{b}{a}; \text{ or, } ma : mb :: a : b; \quad . \quad . \quad . \quad (8)$$

hence, the following principle:

6°. *Equimultiples of two quantities are proportional to the quantities themselves.*

If we multiply both terms of the first member of (1), by m, and both terms of the second member by n, the equality will not be destroyed, and we have,

$$\frac{mb}{ma} = \frac{nd}{nc}; \text{ or, } ma : mb :: nc : nd; \quad . \quad . \quad (9)$$

hence, the following principle:

7°. *If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.*

If, in Equation (8), we suppose $m = 1 \pm \frac{p}{q}$, in which $\frac{p}{q}$ is any fraction, we have the following principle:

8°. *If two quantities be increased or diminished by like parts of each, the results will be proportional to the quantities themselves.*

If, in Equation (9), we suppose, $m = 1 \pm \frac{p}{q}$, and $n = 1 \pm \frac{p'}{q'}$, we have the following principle:

9°. *If both terms of the first couplet of a proportion are increased or diminished by like parts of each, and if both terms of the second couplet are increased or diminished by like parts of each, the results will be in proportion.*

A continued proportion, is one in which several ratios are successively equal to each other; as,

$$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}, \text{ \&c.; or, } a : b :: c : d :: e : f :: g : h, \text{ \&c.,} \quad . \quad . \quad (10)$$

From the preceding continued proportion, we evidently have the following equations:

$$\frac{b}{a} = \frac{b}{a}; \quad \text{whence,} \quad ba = ab.$$

$$\frac{b}{a} = \frac{d}{c}; \quad \text{"} \quad bc = ad.$$

$$\frac{b}{a} = \frac{f}{e}; \quad \text{"} \quad be = af.$$

$$\frac{b}{a} = \frac{h}{g}; \quad \text{"} \quad bg = ah, \text{ \&c.}$$

Adding and factoring, we have,

$$b(a + c + e + g + \text{\&c.}) = a(b + d + f + h + \text{\&c.});$$

changing this equation into a proportion (principle 1°), we have,

$$\frac{b + d + f + h + \text{\&c.}}{a + c + e + g + \text{\&c.}} = \frac{b}{a}; \text{ or,}$$

$$a + c + e + g + \text{\&c.} : b + d + f + h + \text{\&c.} :: a : b; \quad (11)$$

hence, the following principle:

10°. *In any continued proportion, the sum of all the antecedents is to the sum of all the consequents, as any antecedent is to the corresponding consequent.*

Let us assume the two equations,

$$\frac{b}{a} = \frac{d}{c}, \text{ and } \frac{f}{e} = \frac{h}{g};$$

multiplying these equations, member by member, we have,

$$\frac{bf}{ae} = \frac{dh}{cg}; \text{ or, } ae : bf :: cg : dh; \quad . \quad . \quad (12)$$

hence, the following principle:

11°. *If two proportions be multiplied together, term by term, the products will be proportional.*

This principle may be extended to the multiplication of any number of proportions, term by term.

II. SERIES.

Definition.

167. A **series** is a succession of terms, each of which, after a certain number are known, may be derived from one or more of the preceding ones, by a fixed law. This law is called the **law of the series.**

If a certain number of terms are given, and the law of the series is known, any number of terms may be found. There are an infinite number of terms in every series.

The simplest series are, arithmetical and geometrical progressions.

Arithmetical Progression.

168. An **arithmetical progression** is a series in which each term is derived from the preceding term, by adding to it a constant quantity. This quantity is called the **common difference.**

If the common difference is *positive*, each term is greater than the preceding one, and the progression is said to be **increasing.** If the common difference is *negative*, each term is less than the preceding one, and the progression is said to be **decreasing.**

Thus, 2, 4, 6, 8, &c., is an *increasing* arithmetical progression, in which the common difference is $+ 2$.

The series, 18, 16, 14, 12, &c., is a *decreasing* arithmetical progression, in which the common difference is $- 2$.

Although there are an infinite number of terms in every progression, it is customary to speak of a finite number of consecutive ones, as constituting a progression. Thus, we call the succession of terms,

$$3, \quad 5, \quad 7, \quad 9, \quad 11,$$

a progression of 5 terms.

If the terms of any increasing progression are taken in a reverse order, beginning at the last, the result will be a decreasing progression. Thus, the progression,

$$4, \quad 8, \quad 12, \quad 16,$$

becomes, when reversed,

$$16, \quad 12, \quad 8, \quad 4.$$

If a decreasing progression is, in like manner, reversed, the result is an increasing progression.

169. In every arithmetical progression, having a finite number of terms, there are five quantities especially to be considered, viz.: *The first term, the last term, the number of terms, the common difference,* and *the sum of the terms.* When any three of these are given, the other two may be found. In investigating rules for the solution of these different cases, let us denote

the first term by . . . . . . a,
" last term by . . . . . . l,
" number of terms by . . . n,
" common difference by . . d,
" sum of the terms by . . s.

The first and last terms are called **extremes**, all the other terms are called **arithmetical means.**

170. Given a, d, and n, to find l:

The second term is, by definition, equal to $a + d$; the third is equal to the second, increased by d, that is, it is $a + 2d$; the fourth term is equal to the third, increased by d, that is, it is $a + 3d$; and so on: hence, the n^{th} term, or l, is equal to $a + (n - 1)d$; or,

$$l = a + (n - 1)d \quad . \quad . \quad . \quad (1)$$

that is, *any term is equal to the first term,* plus *the product of the common difference by the number of preceding terms.*

EXAMPLES.

1. The first term is 3, and the common difference is 3. What is the 7th term?

Here, the number of terms preceding the 7th, is 6; hence, by the rule, the 7th term is, 3 + 3 × 6, or 21.

2. The first term is 24, and the common difference -3; what is the 5th term?

Ans. $24 + 4 \times -3 = 12$.

3. The first term is 1, and the common difference is $1\frac{1}{2}$; what is the 25th term?

Ans. $1 + 24 \times 1\frac{1}{2} = 37$.

4. The first term is 1, and the common difference $-\frac{3}{4}$; what is the 13th term?

Ans. $1 + 12 \times -\frac{3}{4} = -8$.

5. The first term is -5, and the common difference 2; what is the 7th term?

Ans. $-5 + 2 \times 6 = 7$.

171. Given a, l, and n, to find s:

Having found the n^{th} term, l, the preceding term is equal to $l - d$; the term preceding that, $l - 2d$, and so on. If we write out the terms of the progression, and then write the same terms in a reverse order, the sum will be the same in both cases; hence, we have,

$$s = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l,$$
$$s = l + (l-d) + (l-2d) + \ldots \quad (a+2d) + (a+d) + a;$$

adding these equations, term by term, we have,

$$2s = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l) + (a+l);$$

here, $(a + l)$ is taken n times; hence,

$$2s = n(a + l); \text{ or, } s = \tfrac{1}{2}n(a + l), \quad . \quad . \quad (2)$$

that is, *the sum of the terms is equal to the sum of the extremes, multiplied by half the number of terms.*

Formula (2) can be placed under another form, by substituting for l its value taken from formula (1):

$$s = \frac{1}{2}n(a + a + (n-1)d) = na + \frac{n(n-1)}{2}d, \quad (3)$$

by means of which, the sum of the terms may be found more directly than by formula (2).

EXAMPLES.

1. The first term is 2, the common difference is 3, and the number of terms is 17. What is their sum?

Ans. $17 \times 2 + \frac{17 \times 16}{2} \times 3 = 442.$

2. The first term is $\frac{1}{2}$, the common difference $-\frac{1}{8}$, and the number of terms 20. What is the sum?

Ans. $20 \times \frac{1}{2} + \frac{20 \times 19}{2} \times -\frac{1}{8} = -\frac{55}{4}.$

3. The first term is 20, the common difference is -2, and the number of terms is 6. What is the sum?

Ans. $6 \times 20 + \frac{6 \times 5}{2} \times -2 = 90.$

4. The first term is 5, the common difference 3, and the number of terms 12. What is the sum?

Ans. 258.

5. The first term is -2, the common difference is -3, and the number of terms is 10. What is the sum?

Ans. -155.

Formulas (1) and (2), contain five quantities: a, d, n, l, and s. If any three are assumed at pleasure, the remaining two may be deduced from the formulas.

Geometrical Progression.

172. A **geometrical progression** is a series, each term of which is derived from the preceding one, by multiplying it by a fixed quantity. This quantity is called the **ratio** of the progression.

173. If the first term is *positive*, and the ratio *greater* than 1, each term is greater than the preceding one, and the progression is said to be **increasing**. If the ratio is *less* than 1, each term is less than the preceding one, and the progression is said to be *decreasing*.

Thus, the series 2, 4, 8, 16, &c., is an increasing progression, whose ratio is 2.

The series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., is a decreasing progression, whose ratio is $\frac{1}{2}$.

174. If the ratio is *negative*, the terms of the progression are alternately positive and negative. The positive terms make up a progression whose ratio is equal to the square of the given ratio, and the negative terms make up a second progression, having the same ratio. Thus, the progression $2, -4, +8, -16$, &c., whose ratio is -2, is made up of the two progressions, 2, 8, 32, &c., and -4, -16, -64, &c., whose ratio in each case is 4.

175. In any geometrical progression, there are five quantities to be considered, any three of which being given, the other two may be found. These quantities are,

the *first term*, denoted by . . . . a,
" *last term*, denoted by . . . . l,
" *number of terms*, denoted by . . n,
" *ratio*, denoted by . . . . . . r,
" *sum of the terms*, denoted by . . s.

The first and last terms are called **extremes**; all the other terms are called **geometrical means.**

176. Given a, r, and n, to find l: The second term is, by definition, equal to the first multiplied by r, that is, it is equal to ar; the third term is equal to the second, multiplied by r, that is, it is equal to ar^2; the fourth term is equal to the third, multiplied by r, that is, it is equal to ar^3; and so on to the n^{th} term, which is equal to ar^{n-1}; hence,

$$l = ar^{n-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, *any term of a geometrical progression is equal to the first term, multiplied by that power of the ratio whose exponent is equal to the number of preceding terms.*

EXAMPLES.

1. Find the 7th term of the series 1, 4, 16, &c.

We have, $l = ar^{n-1} = 1 \times 4^6 = 4096$. *Ans.*

2. Find the 8th term of the series 2, 4, 8, &c.

Ans. 256.

3. Find the 12th term of the series 30, 15, $7\frac{1}{2}$, &c.

Ans. $\frac{15}{1024}$.

4. Find the 8th term of the series 5, 25, 125, &c.

Ans. 390625.

177. Given a, r, and n, to find s: we have, from the definition,

$$s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1};$$
$$\therefore\ rs = \qquad ar + ar^2 + \qquad ar^{n-2} + ar^{n-1} + ar^n;$$

subtracting the first of the above equations from the second, member from member, and factoring, we have,

$$s(r-1) = a(r^n - 1);$$
$$\therefore\ s = a\frac{r^n - 1}{r - 1} \quad . \quad . \quad . \quad . \quad (2)$$

Had we subtracted the second from the first, we should have found,

$$s = a\frac{1 - r^n}{1 - r} \quad . \quad . \quad . \quad . \quad (3)$$

If r is greater than 1, formula (2) will be found more convenient; if it is less than 1, formula (3) is to be preferred; but either may be used in any case.

EXAMPLES.

1. Find the sum of 8 terms of the series 5, 20, 80, &c.

$$s = a\frac{r^n - 1}{r - 1} = 5 \cdot \frac{4^8 - 1}{4 - 1} = 109225. \quad \textit{Ans.}$$

2. Find the sum of 7 terms of the series $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, &c.

Ans. $\frac{2059}{1458}$.

3. Find the sum of 6 terms of the series 64, 32, 16, &c. *Ans.* 126.

4. Find the sum of 8 terms of the series 2, -4, $+8$, -16, &c. *Ans.* -170.

178. Solving equation (1), Art. 176, with respect to r, we have,

$$r = \sqrt[n-1]{\frac{l}{a}}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a formula by means of which we can find the *ratio*, when the extremes and the number of terms are given.

The same formula enables us to insert any number of geometrical means between any two numbers. Since the number of means is equal to the whole number of terms diminished by 2, we shall have, $n - 1 = m + 1$, in which m denotes the number of means required; substituting in (4), we have,

$$r = \sqrt[m+1]{\frac{l}{a}}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLES.

1. Insert 4 geometrical means between $\frac{1}{3}$ and 81.

We have, $r = \sqrt[5]{81 \div \frac{1}{3}} = \sqrt[5]{243} = 3$. Hence, the means are, 1, 3, 9, and 27.

2. Insert 3 geometrical means between 2 and $\dfrac{81}{8}$.

Ans. 3, $\dfrac{9}{2}$, and $\dfrac{27}{4}$.

3. Insert 4 geometrical means between 2 and 486.

Ans. 6, 18, 54, 162.

4. Insert 5 geometrical means between 1 and $\dfrac{1}{64}$.

Ans. $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}$.

5. Insert 4 geometrical means between 2 and 6250.

Ans. 10, 50, 250, 1250.

Formula (3), Art. 177, may be placed under the form,

$$s = \frac{a}{1-r} - \frac{ar^n}{1-r} \quad . \quad . \quad . \quad (6)$$

If r is a fraction less than 1, and if n is very great, the numerator, ar^n, is very small with respect to the denominator; and finally, if $n = \infty$, the value of ar^n is 0. In this case, the value of s reduces to its first term, and we have,

$$s = \frac{a}{1-r} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

179. In every decreasing progression, the value, $\frac{a}{1-r}$, is that towards which the sum of the series approaches, as the number of terms is increased. This value is called, the *limit, or sum of the progression.*

EXAMPLES.

1. Find the sum of the terms, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, &c., to infinity.

We have, $s = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 4.$ *Ans.*

2. Find the sum of the terms, $\frac{5}{3}$, 1, $\frac{3}{5}$, &c., to infinity. *Ans.* $4\frac{1}{6}$.

3. Find the sum of the terms, 1, $\frac{1}{3}$, $\frac{1}{9}$, &c., to infinity. *Ans.* $\frac{3}{2}$.

4. Find the sum of the terms, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{8}$, &c., to infinity. *Ans.* $2\frac{2}{3}$.

5. What is the limit towards which the sum of the series 1, $\frac{4}{5}$, $\frac{16}{25}$, &c., approaches, as the number of terms increases? *Ans.* 5.

6. What is the limit of the sum of the series $1,\ \frac{3}{4},\ \frac{9}{16}$, &c.? *Ans.* 4.

PROBLEMS.

1. The sum of three terms in geometrical progression is 21, and the sum of their reciprocals is $\frac{7}{12}$. What are the terms?

Denote the first term by x, and the ratio by y; we shall have, from the conditions of the problem,

$$\left.\begin{aligned} x + xy + xy^2 &= 21 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \\ \frac{1}{x} + \frac{1}{xy} + \frac{1}{xy^2} &= \frac{7}{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}\right\}$$

From (1) and (2), we find,

$$1 + y + y^2 = \frac{21}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$1 + y + y^2 = \frac{7}{12}xy^2 \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore \quad \frac{21}{x} = \frac{7}{12}xy^2; \text{ or, } x^2y^2 = 36; \text{ or, } xy = 6;$$

substituting the value of xy in (4), we have,

$$1 + y + y^2 = \frac{7}{12} \times 6y.$$

$$\therefore \quad y^2 - \frac{5}{2}y = -1;$$

$$\text{or,} \quad y = \frac{5}{4} \pm \sqrt{\frac{25}{16} - 1};$$

using the positive value of y only, we have,

$$y = 2; \quad \text{whence,} \quad x = 3.$$

The terms are, therefore, 3, 6, and 12.

2. The population of a town increases annually in a geometrical ratio, and in three years the population

rises from 120,000 to 138,915; by what part of itself is it increased each year?

Let x denote the ratio of increase; from the conditions of the problem,

$$120000x^3 = 138915;$$

or,

$$8000x^3 = 9261.$$

$$\therefore \quad 20x = 21; \text{ or, } x = 1\tfrac{1}{20};$$

hence, each year, there is added one-twentieth of its population.

3. The sum of a geometrical progression to an infinite number of terms is 2, and the sum of the squares of the terms of the same series is $\frac{4}{3}$. What is the first term, and the ratio of the given progression?

Let x denote the first term, and y the ratio; we then have,

$$x + xy + xy^2 + \&c. = \frac{x}{1-y} = 2; \quad . \quad . \quad . \quad (1)$$

$$x^2 + x^2y^2 + x^2y^4 + \&c. = \frac{x^2}{1-y^2} = \frac{4}{3}; \quad . \quad . \quad . \quad (2)$$

dividing (2) by (1), we have,

$$\frac{x}{1+y} = \frac{2}{3}; \text{ or, } 3x = 2 + 2y; \quad . \quad . \quad . \quad . \quad (3)$$

from (1), we have,

$$x = 2 - 2y; \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

from (3) and (4), by combination,

$$x = 1, \text{ and } y = \tfrac{1}{2};$$

the series is, therefore,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \&c.$$

4. The population of a town increases annually in geometrical progression, and in four years is raised from 10,000 to 14,641; by what part of itself is it increased each year? *Ans.* $\frac{1}{10}$.

5. Find 4 numbers in geometrical progression, such that the sum of the means is 36, and the sum of the extremes 84. *Ans.* 3, 9, 27, and 81.

6. Insert 3 geometrical means between $\frac{1}{2}$ and $\frac{81}{32}$.
Ans. $\frac{3}{4}$, $\frac{9}{8}$, $\frac{27}{16}$.

III. INDETERMINATE COEFFICIENTS.

Definitions and Explanations.

180. An **identical equation** is an equation that is true for all values of the unknown quantities that enter it. Thus,

$$ax + b = ax + b, \quad \text{and} \quad \frac{a^2 - x^2}{a - x} = a + x,$$

are identical equations.

Every identical equation containing but one unknown quantity, can be reduced to the form of,

$$p + qx + rx^2 + \&c. = p' + q'x + r'x^2 + \&c. \quad (1)$$

or, by transposition, to the form,

$$(p - p') + (q - q')x + (r - r')x^2 + \&c. = 0. \quad (2)$$

Because equations (1) and (2) are true for all values of x, it follows that x is indeterminate (Art. 89); the coefficients of the different powers of x are therefore coefficients of indeterminate quantities, and for this reason they are called **indeterminate coefficients**.

Principle of Indeterminate Coefficients.

181. Since equation (1) is true for all values of x, it must be true when $x = 0$; making $x = 0$ in that equation, we have, $p = p'$. Suppressing p in the first member of equation (1) and its equal p' in the second member, and then dividing both members by x, we have,

$$q + rx + \&c. = q' + r'x + \&c. \quad . \quad . \quad (3)$$

which is an identical equation because it is only a modified form of equation (1); hence, equation (3) is true for all values of x.

Making $x = 0$ in (3), we have as before $q = q'$. Suppressing q and q' in (3), dividing by x and then making $x = 0$ in the resulting equation, we have $r = r'$, and so on indefinitely. Hence, we have the following principle:

In any identical equation, containing but one indeterminate quantity, the coefficients of the like powers of this quantity in the two members, are separately equal to each other.

If all the terms of an identical equation are transposed to one member, as in equation (2), article 180, the coefficients of the different powers of the unknown quantity are separately equal to 0.

Extension of the Preceding Principle.

182. If an identical equation contains two or more indeterminate quantities, it may be shown by a similar course of reasoning, that the coefficients of the like powers and combinations of powers of these quantities, in the two members, are separately equal to each other; hence, the following general principle:

In any identical equation containing any number of indeterminate quantities, the coefficients of the like powers and combinations of powers of these quantities, in the two members, are separately equal to each other.

Application to Series.

183. The principle of indeterminate coefficients may be used in developing algebraic expressions into series. To illustrate the method of proceeding, let it be required to develop the expression $\frac{2 + 3x}{3 + 4x + 5x^2}$ into a series arranged with respect to the ascending powers of x:

Assume the equation,

$$\frac{2 + 3x}{3 + 4x + 5x^2} = p + qx + rx^2 + sx^3 + \&c. \quad (1)$$

and suppose it identical; clearing of fractions, we have,

$$2 + 3x = \begin{array}{r|r|r|l} 3p + 4p & x + 5p & x^2 + 5q & x^3 + \&c.; \\ + 3q & + 4q & + 4r & \\ & + 3r & + 3s & \end{array}$$

from the principle of indeterminate coefficients, we have,

$$\begin{array}{llll} 2 = 3p, & \therefore\ p & = & \frac{2}{3}. \\ 3 = 4p + 3q, & \therefore\ q & = & \frac{1}{9}. \\ 0 = 5p + 4q + 3r, & \therefore\ r & = & -\frac{34}{27}. \\ 0 = 5q + 4r + 3s, & \therefore\ s & = & \frac{121}{81},\ \&c.; \end{array}$$

substituting these values in (1), we have the required series,

$$\frac{2+3x}{3+4x+5x^2} = \frac{2}{3} + \frac{1}{9}x - \frac{34}{27}x^2 + \frac{121}{81}x^3, \text{ \&c.}$$

In like manner, any similar expression may be developed into a series; hence, the following

RULE.

Assume the given expression equal to a series of the form, $p + qx + rx^2 + $ &c., in which p, q, r, &c., are quantities to be determined; clear the equation of fractions, and place the coefficients of the like powers of the unknown quantity in the two members separately equal to each other; then find, from the resulting equations, the values of p, q, r, &c., and substitute these values in the assumed development.

EXAMPLES.

1. Develop $\dfrac{1-x}{1+x}$, into a series.

OPERATION.

$$\frac{1-x}{1+x} = p + qx + rx^2 + sx^3 + \text{\&c.};$$

clearing of fractions,

$$1 - x = \left.\begin{matrix} p + q \\ +p \end{matrix}\right| x + \left.\begin{matrix} r \\ +q \end{matrix}\right| x^2 + \left.\begin{matrix} s \\ +r \end{matrix}\right| x^3 + \text{\&c.};$$

equating coefficients,

$$\begin{aligned} 1 &= p, & \therefore\ p &= 1. \\ -1 &= q + p, & \therefore\ q &= -2. \\ 0 &= r + q, & \therefore\ r &= 2. \\ 0 &= s + r, & \therefore\ s &= -2, \text{ \&c.}; \end{aligned}$$

substituting, we have,

$$\frac{1-x}{1+x} = 1 - 2x + 2x^2 - 2x^3, \text{\&c.}$$

The law of the series is evident, and any number of terms may be written out by means of this law.

8. Develop $\dfrac{1-x}{1+x+x^2}$, into a series.

OPERATION.

$$\frac{1-x}{1+x+x^2} = p + qx + rx^2 + sx^3 + \text{\&c.};$$

clearing of fractions, we have,

$$1 - x = p + \left.\begin{array}{r} q \\ +p \\ \end{array}\right| x + \left.\begin{array}{r} r \\ +q \\ +p \end{array}\right| x^2 + \left.\begin{array}{r} s \\ +r \\ +q \end{array}\right| x^3 + \text{\&c.};$$

equating coefficients,

$$\begin{array}{rll} 1 &= p, & \therefore\ p = 1. \\ -1 &= q + p, & \therefore\ q = -2. \\ 0 &= r + q + p, & \therefore\ r = 1. \\ 0 &= s + r + q, & \therefore\ s = 1, \text{\&c.}; \end{array}$$

substituting, we have,

$$\frac{1-x}{1+x+x^2} = 1 - 2x + x^2 + x^3 - 2x^4 + x^5 + x^6 - 2x^7, \text{\&c.}$$

3. Develop $\dfrac{1+2x}{1-3x}$, into a series.

Ans. $1 + 5x + 15x^2 + 45x^3 + 135x^4 + \text{\&c.}$

4. Develop $\dfrac{1+2x}{1-x-x^2}$, into a series.

Ans. $1 + 3x + 4x^2 + 7x^3 + \text{\&c.}$

If the numerator of the fraction contains the unknown quantity to a higher power than the denominator, we first reduce it to a mixed quantity, in which the numerator of the fractional part is of a less degree than the denominator, and then develop the fractional part, by the rule, annexing the result to the entire part.

5. Develop $\dfrac{1 + ax^2 - x^3}{a - x}$, into a series.

Reducing to a mixed quantity,

$$\frac{1 + ax^2 - x^3}{a - x} = x^2 + \frac{1}{a - x};$$

developing, by the rule,

$$\frac{1}{a - x} = \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \&c.$$

$$\therefore \quad \frac{1 + ax^2 - x^3}{a - x} = x^2 + \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \&c.$$

Application to Partial Fractions.

184. If the denominator of a fraction can be resolved into factors of the first degree, the fraction itself can be resolved into partial fractions having these factors for denominators, by the principle of indeterminate coefficients. The preceding remark is applicable here, namely, when the degree of the numerator is higher than that of the denominator, the fraction must be changed to a mixed quantity, as before. The method of proceeding is indicated in the following example:

1. Resolve the fraction, $\dfrac{2a^2}{a^2 - x^2}$, into partial fractions:

OPERATION.

Assume, $\dfrac{2a^2}{a^2 - x^2}, = \dfrac{p}{a + x} + \dfrac{q}{a - x}$, in which p and q are to be determined; clearing of fractions,

$$2a^2 = pa - px + qa + qx;$$

equating the coefficients of the like powers of x,

$$\left.\begin{aligned} 2a^2 &= pa + qa \\ 0 &= -p + q \end{aligned}\right\};$$

whence, $q = a$, and $p = a$;

substituting in the assumed equation,

$$\frac{2a^2}{a^2 - x^2} = \frac{a}{a - x} + \frac{a}{a + x}.$$

2. Resolve $\frac{x^2 - x + 2}{(x-1)(x-2)(x-3)}$. into partial fractions:

OPERATION.

Assume,

$$\frac{x^2 - x + 2}{(x-1)(x-2)(x-3)} = \frac{p}{x-1} + \frac{q}{x-2} + \frac{r}{x-3};$$

clearing of fractions,

$$x^2 - x + 2 = p(x^2 - 5x + 6) + q(x^2 - 4x + 3) + r(x^2 - 3x + 2)$$

equating coefficients, we have,

$$\left.\begin{aligned} 2 &= 6p + 3q + 2r \\ -1 &= -5p - 4q - 3r \\ 1 &= p + q + r. \end{aligned}\right\}$$

$$\therefore\ p = 1,\quad q = -4,\quad \text{and}\quad r = 4.$$

substituting in the assumed equation, we have,

$$\frac{x^2 - x + 2}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{4}{x-2} + \frac{4}{x-3}.$$

3. Resolve the fraction, $\frac{3x - 5}{x^2 - 6x + 8}$, into partial fractions.

Ans. $\frac{7}{2}\cdot\frac{1}{x-4} - \frac{1}{2}\cdot\frac{1}{x-2}.$

4. Resolve $\dfrac{5-x}{1-x^2}$, into partial fractions.

$$Ans.\ \frac{2}{1-x}+\frac{3}{1+x}.$$

5. Resolve $\dfrac{3-2x-4x^2}{(1+2x)(1-2x)(1-x)}$, into partial fractions.

$$Ans.\ \frac{1}{1+2x}+\frac{1}{1-2x}+\frac{1}{1-x}.$$

If the denominator is of the second degree, it can always be resolved into two factors of the first degree, by placing it equal to 0, and finding the roots of the equation; then, the factors will be found by subtracting each root from the unknown quantity.

We have hitherto supposed that the factors of the denominator are unequal. When some of the factors are equal, we proceed as in the following example:

6. Resolve $\dfrac{3x^2-7x+6}{(x-1)^3}$, into partial fractions.

Assume, $\dfrac{3x^2-7x+6}{(x-1)^3}=\dfrac{p}{(x-1)^3}+\dfrac{q}{(x-1)^2}+\dfrac{r}{x-1}$;

clearing of fractions,

$$3x^2-7x+6=p+q(x-1)+r(x^2-2x+1);$$

equating coefficients

$$\left.\begin{aligned} 6 &= p-q+r \\ -7 &= q-2r \\ 3 &= r \end{aligned}\right\}$$

$$\therefore\ p=2,\quad q=-1,\quad \text{and}\quad r=3.$$

substituting in the assumed equation, we have,

$$\frac{3x^2-7x+6}{(x-1)^3}=\frac{2}{(x-1)^3}-\frac{1}{(x-1)^2}+\frac{3}{x-1}.$$

7. Resolve $\frac{x}{(x-1)^2}$, into partial fractions.

$$Ans.\quad \frac{1}{(x-1)^2}+\frac{1}{x-1}.$$

8. Resolve $\frac{x+2x^2}{(x-1)^2}$, into partial fractions.

$$Ans.\quad \frac{3x}{(x-1)^2}+\frac{2x}{(x-1)}.$$

9. Resolve $\frac{1-x+x^2}{(x-1)^3}$, into partial fractions.

$$Ans.\quad \frac{1}{(x-1)^3}+\frac{1}{(x-1)^2}+\frac{1}{x-1}.$$

10. Resolve $\frac{x^2}{(x^2-1)(x-2)}$, into partial fractions.

$$Ans.\quad \frac{4}{3(x-2)}-\frac{1}{2(x-1)}+\frac{1}{6(x+1}.$$

CHAPTER XI.

LOGARITHMS.

Definitions.

185. The **logarithm** of a number is the exponent of the power to which it is necessary to raise a fixed number to produce the given number. The fixed number is called the **base of the system.**

186. If we denote any positive number, except 1, by a, any positive number whatever by n, and the exponent of the power to which it is necessary to raise a, in order to produce n, by x, we shall have the exponential equation,

$$a^x = n \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation, a is the *base*, n any positive number, and x is the logarithm of n. It is plain, that a cannot be negative, neither can it be equal to 1, because every power of 1 is equal to 1.

If, whilst a remains fixed in value, we suppose n to assume in succession every value from 0 to ∞, the corresponding values of x, taken together, will constitute what is called a **system of logarithms.** Since there are an infinite number of different values that may be attributed to a, it follows, that *there are an infinite number of systems of logarithms.* Of these, two only are in general use, viz.: the system whose base is 10, called the **common system**; and the system whose base is 2.718281828..., called the **Napierian system.**

In what follows, we shall designate common logarithms by the symbol *log*, Napierian logarithms by the symbol *l*, and logarithms taken in any system whatever, by the symbol *Log*.

187. If we make $a = 10$, in equation (1), we have the equation,

$$10^x = n \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If n is made equal to 1, in equation (2), the corresponding value of x is 0; if n is made equal to 10, the corresponding value of x is 1; if n is made equal to 100, the corresponding value of x is 2; and so on; hence, we have, from what precedes,

$$\begin{aligned} \log \quad 1 &= 0, \\ \log \quad 10 &= 1, \\ \log \quad 100 &= 2, \\ \log 1000 &= 3, \ \&c. \end{aligned}$$

For all values of n between 1 and 10, the corresponding logarithms lie between 0 and 1; that is, they are fractions less than 1, and are generally expressed decimally. For all values of n between 10 and 100, the corresponding logarithms lie between 1 and 2; that is, they are equal to 1 *plus* a decimal. The logarithms of all numbers between 100 and 1000, lie between 2 and 3; that is, they are equal to 2 *plus* a decimal.

In general, a logarithm is composed of two parts: *an entire part*, called the **characteristic**; and a *decimal part*, sometimes called the **mantissa**.

Logarithms are used to facilitate numerical computations, where they serve to convert operations of multiplication and division into the simpler ones of addition and subtraction. The following principles indicate the methods of applying logarithms to arithmetical computations.

Principles of Logarithms.

188. Let a denote the base of any system of logarithms, m and n any two numbers, and x and y their logarithms. We have, from equation (1),

$$a^x = m \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$a^y = n; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

multiplying (3) and (4), member by member, we have,

$$a^{x+y} = mn;$$

whence, from the definition,

$$x + y = \text{Log } mn; \quad . \quad . \quad . \quad (5)$$

hence, the following principle:

1°. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the two numbers.*

If we divide (3) by (4), member by member, we have,

$$a^{x-y} = \frac{m}{n};$$

whence, from the definition,

$$x - y = \text{Log}\,\frac{m}{n}; \quad . \quad . \quad . \quad . \quad (6)$$

hence, the following principle:

2°. *The logarithm of the quotient is equal to the logarithm of the dividend diminished by that of the divisor.*

If we raise both members of (3) to any power denoted by p, we have,

$$a^{px} = m^p;$$

whence, by definition,

$$px = \text{Log}\,m^p; \quad . \quad . \quad . \quad . \quad (7)$$

hence, the following principle:

3°. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

If we extract any root of both members of (3), denoted by r, we have,

$$a^{\frac{x}{r}} = \sqrt[r]{m};$$

whence, by definition,

$$\frac{x}{r} = \text{Log}\,\sqrt[r]{m} \quad . \quad . \quad . \quad . \quad (8)$$

hence, the following principle:

4°. *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

The applications of the above principles require a table of logarithms. **A table of logarithms**, is a table by means of which the logarithm corresponding to any number, or the number corresponding to any logarithm, may be found.

The principles above demonstrated, give rise to four practical

RULES.

1°. To find the product of two or more numbers:

Find the logarithms of the factors from a table, and take their sum; then find the number corresponding to the resulting logarithm, and it will be the product required.

2°. To find the quotient of one number by another:

Find the logarithms of the dividend and divisor from a table, and subtract the latter from the former; then find the number corresponding to the resulting logarithm, and it will be the quotient required.

3°. To raise a number to any power.

Find the logarithm of the number from a table, and multiply it by the exponent; then find the number corresponding to the resulting logarithm, and it will be the power required.

4°. To extract any root of a number.

Find the logarithm of the number from a table, and divide it by the index; then find the number *corresponding to the resulting logarithm, and it will be the root required.*

No practical examples can be given to illustrate the preceding rules, without a table of logarithms. A few examples of transformation are annexed, which show the methods of proceeding in the employment of logarithms.

EXAMPLES.

1. Transform the equation, $x = \frac{abc}{df}$.

From equations (5) and (6), using common logarithms, we have,

$$\log x = \log a + \log b + \log c - \log d - \log f.$$

2. Transform the equation, $x = \frac{a^7 b^8}{\sqrt[3]{a^2}}$.

From equations (5), (6), (7), and (8), we have,

$$\log x = 7 \log a + 8 \log b - \tfrac{2}{3} \log a = 6\tfrac{1}{3} \log a + 8 \log b.$$

3. Transform the equation, $x = \frac{a^{\frac{2}{3}} b^4 c^{\frac{1}{3}}}{\sqrt[3]{fg}}$.

Ans. $\log x = \frac{2}{3} \log a + 4 \log b + \frac{1}{3} \log c - \frac{1}{3} \log f - \frac{1}{3} \log g.$

4. Transform the equation, $x = \frac{\sqrt{5} \times \sqrt[3]{6}}{\sqrt[7]{3} \times \sqrt[9]{2}}$.

Ans. $\log x = \frac{1}{2} \log 5 + \frac{1}{3} \log 6 - \frac{1}{7} \log 3 - \frac{1}{9} \log 2.$

5. Transform the equation, $x = \frac{\sqrt[3]{(a^2 - b^2) \times 3a}}{\sqrt{(a + b)c^3}}$.

Ans. $\log x = \frac{1}{3} \log (a - b) + \frac{1}{3} \log a + \frac{1}{3} \log 3 - \frac{1}{6} \log (a + b) - \frac{3}{2} \log c.$

Solve the following equations:

6. $7^x = 13.$

Taking the logarithms of both members, we have,

$$x \log 7 = \log 13. \qquad \therefore \quad x = \frac{\log 13}{\log 7}. \quad \textit{Ans.}$$

7. $\left(\frac{5}{7}\right)^x = \frac{2}{3}.$ *Ans.* $x = \frac{\log 2 - \log 3}{\log 5 - \log 7}.$

8. $ab^x = c.$ *Ans.* $x = \frac{\log c - \log a}{\log b}.$

9. $3^{\sqrt{x}} = 5.$ *Ans.* $x = \left(\frac{\log 5}{\log 3}\right)^2.$

10. $\left.\begin{aligned} a^x b^y &= c \\ my &= nx \end{aligned}\right\}$

Taking the logarithms of both members of the first equation, we have,

$$x \log a + y \log b = \log c.$$

Combining this with the second of the given equations, we find,

$$x = \frac{m \log c}{m \log a + n \log b}, \text{ and } y = \frac{n \log c}{m \log a + n \log b}.$$

11. $(a^2 - b^2)^{2(x-1)} = (a - b)^{2x}.$

Ans. $x = 1 + \frac{\log (a - b)}{\log (a + b)}.$

General Properties of Logarithms.

189. There are certain general properties of logarithms that may be discovered by a discussion of the exponential equation,

$$a^x = n \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation, the arbitrary quantities are a and n.

1°. If we make $n = 1$, the corresponding value of x will be 0, whatever may be the value of a, since $a^0 = 1$; hence,

The logarithm of 1, *in any system, is equal to* 0.

2°. If we make $n = a$, the corresponding value of x will be 1, whatever may be the value of a; hence,

The logarithm of the base of any system, taken in that system, is 1.

3°. If we suppose $a > 1$, say 10, for example, we shall have,

$$10^x = n.$$

If $n = 1$, the value of x, or the logarithm of 1, is 0; if $n = \infty$, the value of x, or the logarithm of ∞, is ∞. The logarithms of all numbers between 1 and ∞, lie between 0 and ∞, that is, they are positive.

If n is less than 1, x must be negative, giving $\frac{1}{10^x} = n$; if $n = 0$, x will be infinite, in the last equation, because $\frac{1}{10^\infty} = 0$, therefore, $x = -\infty$, in the given equation, that is, the logarithm of 0 is equal to $-\infty$; hence,

In any system whose base is greater than 1, *the logarithms of all numbers greater than* 1, *are positive; the logarithms of all numbers less than* 1, *are negative; the logarithm of* ∞, *is* $+\infty$, *and the logarithm of* 0 *is* $-\infty$.

4°. If we suppose $a < 1$, say $\frac{1}{10}$, for example, we shall have,

$$\frac{1}{10^x} = n, \quad \text{or} \quad 10^{-x} = n;$$

in this case, the positive values of x correspond to the negative values of x in the preceding case; and the negative values of x, to the positive values, in the preceding case; hence,

In any system whose base is less than 1, *the logarithms of all numbers greater than* 1, *are negative; the logarithms of all numbers less than* 1, *are positive; the logarithm of* ∞, *is* $-\infty$, *and the logarithm of* 0, *is* $+\infty$.

5°. Since, for every value of x between $-\infty$ and $+\infty$, that is, for every real value of x, the values of n lie between 0 and $+\infty$, whether a is greater or less than 1, it follows that there are no real values of x, which, substituted in the equation, $a^x = n$, will make n negative; hence,

There are no real logarithms corresponding to negative numbers.

Although there are no logarithms of negative numbers, we may multiply negative numbers by means of logarithms. We first regard the numbers as positive; and, having applied the rules, we then give the proper sign to the result, according to the rule for signs. Thus, to multiply 27 by -435, we find the product of 27 and 435, and give it the minus sign.

Logarithmic Series.

190. Let it be required to develop Log $(1 + y)$ into a series arranged according to the ascending powers of y. If we make $y = 0$ in the expression Log $(1 + y)$, it reduces to Log 1, which is equal to 0; hence, the series must be such that it will reduce to 0 when $y = 0$, that is, every term must contain y as a factor. We may therefore assume the series,

$$\text{Log}\,(1 + y) = My + Ny^2 + Py^3 + Qy^4 + \text{\&c.} \quad (1)$$

in which M, N, P, &c., are constants to be determined. Since equation (1) must be true for all values of y, we may write z for y, giving the equation,

$$\text{Log}\,(1 + z) = Mz + Nz^2 + Pz^3 + Qz^4 + \text{\&c.} \quad (2)$$

Subtracting (2) from (1), member from member, and remembering that

$$\text{Log}\,(1+y) - \text{Log}\,(1+z) = \text{Log}\left(\frac{1+y}{1+z}\right) = \text{Log}\left(1+\frac{y-z}{1+z}\right)$$

we have,

$$\text{Log}\left(1+\frac{y-z}{1+z}\right) = M(y-z) + N(y^2-z^2) + P(y^3-z^3) + \text{\&c.} \quad (3)$$

Every term of the second member of (3) is divisible by $y - z$ by means of formula 9, Art. 36.

Writing $\frac{y-z}{1+z}$ for y in equation (1), we have,

$$\text{Log}\left(1+\frac{y-z}{1+z}\right) = M\left(\frac{y-z}{1+z}\right) + N\left(\frac{y-z}{1+z}\right)^2 + P\left(\frac{y-z}{1+z}\right)^3 + \text{\&c.} \quad (4)$$

Placing the second member of (4) equal to the second member of (3), and dividing both by $y - z$, we have,

$$\frac{M}{1+z} + N\frac{y-z}{(1+z)^2} + P\frac{(y-z)^2}{(1+z)^3} + \&c. = M + N(y+z)$$
$$+ P(y^2 + yz + z^2) + \&c.$$

Since this equation is true for all values of y and z, make $z = y$, and we shall have,

$$\frac{M}{1+y} = M + 2Ny + 3Py^2 + 4Qy^3 + \&c. \quad . \quad (5)$$

which is an identical equation; clearing of fractions, we have,

$$M = \begin{array}{r|r|r|l} M + 2N & y + 3P & y^2 + 4Q & y^3 + \&c. \\ + \ M & + 2N & + 3P & \end{array}$$

Making the coefficients of the like powers of y in the two members equal (Art. 178), we have,

$$\begin{array}{lll} M = M & & \\ 0 = 2N + M, & \therefore \quad N = -\dfrac{M}{2}, & \\ 0 = 3P + 2N, & \therefore \quad P = -\dfrac{2N}{3} = & \dfrac{M}{3}, \\ 0 = 4Q + 3P, & \therefore \quad Q = -\dfrac{3P}{4} = & -\dfrac{M}{4}, \\ \quad \&c., & \qquad \&c. & \end{array}$$

Substituting these values of N, P, Q, &c., in equation (1) and factoring, we have,

$$\text{Log}\,(1+y) = M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \&c.,\right) \quad (6)$$

which is the logarithmic series.

The quantity M is a constant that depends on the base of the system. It is called the **modulus.**

In the Napierian system $M = 1$, and the logarithmic series in this case becomes

$$l(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \&c. \quad . \quad (7)$$

This series is not suitable for computing the logarithms of numbers. To deduce a series for this purpose, we write $-y$ for y in equation (7), which gives,

$$l(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \&c. \quad . \quad (8)$$

Subtracting (8) from (7), member from member, remembering that $l(1+y) - l(1-y)$ is equal to $l\left(\frac{1+y}{1-y}\right)$, we have,

$$l\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \&c.\right) \quad . \quad (9)$$

If we now make $\frac{1+y}{1-y} = \frac{z+1}{z}$, whence $y = \frac{1}{2z+1}$, we have,

$$l\left(\frac{z+1}{z}\right) = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right)$$

Replacing $l\left(\frac{z+1}{z}\right)$ by its equal $l(z+1) - lz$, and transposing lz to the second member, we have,

$$l(z+1) = lz + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) \quad . \quad . \quad . \quad . \quad (10)$$

This formula enables us to compute $l(z+1)$ when we know the value of lz.

CHAPTER XII.

GENERAL THEORY OF EQUATIONS.

I. PROPERTIES AND TRANSFORMATIONS.

General Form.

191. Any equation containing but one unknown quantity, and whose exponents are whole numbers, may be reduced to the form,

$$x^n + px^{n-1} + qx^{n-2} + \&c. + sx^2 + tx + u = 0 \quad (1).$$

In this equation, n is a positive whole number, but the coefficients p, q, &c., may be either positive or negative, entire or fractional, real or imaginary. The method of reducing equations to the form (1), is analogous to that given for reducing equations of the second degree to the form,

$$x^2 + 2px = q;$$

and since the reduction can always be made, we shall hereafter, in speaking of equations, suppose them reduced to the form of equation (1), unless the contrary is expressly mentioned.

Roots.

192. Any value of x, either real or imaginary, which, if substituted for x in equation (1) will satisfy it, that is, make the two members equal, is a **root** of the equation. It has been shown that every equation of the first degree has *one root*, and that every equation of the second degree has *two roots;* we shall assume that every equation of the n^{th} degree has at least one root, either real, or imaginary.

Properties and Transformations.

193. In this and the following articles it is proposed to demonstrate the most important properties and transformations of equation (1), article 191.

First Property.

If a *is a root of equation* (1), *the first member of that equation is divisible by* $x - $ a.

For, if we divide the first member of equation (1) by $x - a$, and continue the division till a remainder is found that does not contain x, and if we denote that remainder by n and the quotient obtained by m, we have,

$$x^n + px^{n-1} + \&c. + tx + u = (x - a)m + n. \quad (2).$$

Now, if a is a root of the proposed equation, it will reduce the first member of (2) to 0, when substituted for x; it will also reduce the first term of the second member to 0; hence, n is also equal to 0, that is, the remainder is 0, and consequently the first member is exactly divisible by $x - a$, which was to be shown.

Second Property.

194. *If the first member of equation* (1) *is exactly divisible by* $x - a$, *then is* a *a root of the equation.*

If we divide the first member of equation (1) by $x - a$, as explained in the last article, the remainder n will be equal to 0, and equation (2) will reduce to the form,

$$x^n + px^{n-1} + \&c. + tx + u = (x - a)m. \quad (3).$$

If, in (3), we make $x = a$, the second member reduces to 0; consequently, the first member also reduces to 0, which satisfies equation (1); hence, a is a root of (1), which was to be shown.

It follows from the preceding propositions that we can ascertain whether a polynomial containing x, is exactly divisible by $x - a$, by substituting a for x in the polynomial: if the result is 0, the polynomial is divisible by $x - a$; if not, the polynomial is not divisible by $x - a$.

Third Property.

195. *Equation* (1) *has as many roots as there are units in* n, *and it has no more.*

It is assumed that the equation has one root; let that root be denoted by a: then will $x - a$ be a factor of the first member, and the first term of the other factor will be x^{n-1}; the exponents of x in the succeeding terms of the second factor will be less than

$n-1$; hence, equation (1) may be written under the form,

$$(x-a)(x^{n-1}+p'x^{n-2}+\&c.+t'x+u')=0 \quad . \quad (4).$$

Now, equation (4) may be satisfied in two ways, viz.: by placing the first factor equal to 0, or by placing the second factor equal to 0. In the latter case, we have the equation,

$$x^{n-1}+p'x^{n-2}+\&c.+t'x+u'=0 \quad . \quad . \quad . \quad (5).$$

Now, equation (5) has at least one root; let that root be designated by b; then it may be shown as before, that equation (5) can be written under the form,

$$(x-b)(x^{n-2}+p''x^{n-3}+\&c.+t''x+u'')=0; \quad . \quad (6)$$

which reduces equation (4) to the form of,

$$(x-a)(x-b)(x^{n-2}+p''x^{n-3}+\&c.+t''x+u'')=0 \quad . \quad (7).$$

In the same manner as before, it may be shown that the second factor of the first member of (6) can be placed equal to 0, and factored, giving

$$(x-c)(x^{n-3}+p'''x^{n-4}+\&c.+t'''x+u''')=0;$$

which reduces equation (4) to the form,

$$(x-a)(x-b)(x-c)(x^{n-3}+p'''x^{n-4}+\&c.+t'''x+u''')=0.$$

By continuing the process, it may be shown that the first member will ultimately be resolved into just as many binomial factors, of the form, $(x-a)$, $(x-b)$, &c.,

as there are units in n. Hence, equation (1) may be written under the form,

$$(x-a)(x-b)(x-c)\ldots(x-k)(x-l) = 0. \quad . \quad . \quad (8).$$

Equation (8) may be satisfied, by placing either of the factors, $x - a$, $x - b$, &c., equal to 0, and either factor being placed equal to 0, gives a root. Now, since there are n factors, the equation has n roots, and since the equation cannot be satisfied except by making one of the factors equal to 0, there are only n roots; which was to be shown.

It is to be observed that some of the roots, and consequently some of the binomial factors, may be equal to each other.

Applications of the Third Property.

196. If both members of equation (8) are divided by any one of its binomial factors, the resulting equation will be freed from the corresponding root.

EXAMPLES.

1. One root of the equation, $x^3 - 9x^2 + 26x - 24 = 0$, is 4; what does the equation become when freed of this root?

OPERATION.

$$\begin{array}{r|l} x^3 - 9x^2 + 26x - 24 & x - 4 \\ \cline{2-2} \underline{x^3 - 4x^2 \qquad\qquad} & x^2 - 5x + 6 \\ -5x^2 + 26x \qquad & \\ \underline{-5x^2 + 20x \qquad} & \\ 6x - 24 & \\ \underline{6x - 24} & \\ 0 & \end{array}$$

hence, the required equation is,

$$x^2 - 5x + 6 = 0,$$

which may be solved by known rules.

2. One root of the equation, $x^3 - 37x - 84 = 0$, is $+ 7$; what does it become when freed of this root?
Ans. $x^2 + 7x + 12 = 0$.

3. One root of the equation, $x^3 - 11x^2 + 16x + 84 = 0$, is $- 2$; what does the equation become when freed of this root? *Ans.* $x^2 - 13x + 42 = 0$.

4. One root of the equation, $x^3 + 7x^2 - 4x - 28 = 0$, is $- 7$; what does the equation become when freed of this root? *Ans.* $x^2 - 4 = 0$.

5. One root of the equation, $x^3 - 12x^2 + 47x - 60 = 0$, is 3; what are the other roots? *Ans.* 4 and 5,

6. One of the roots of $x^3 + 9x^2 + 26x + 24 = 0$, is $- 4$; what are the other two? *Ans.* $- 2$ and $- 3$.

7. Two roots of the equation, $x^4 - 12x^3 + 48x^2 - 68x + 15 = 0$, are 3 and 5; what are the other two?
Ans. $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

8. One root of the equation, $x^3 - 6x^2 + 11x - 6 = 0$, is 1; what are the other two? *Ans.* 3 and 2.

From equation (8), we deduce the following rule for forming an equation whose roots are given:

RULE.

Subtract each root from the unknown quantity; multiply the resulting binomials together, and place the product equal to 0.

EXAMPLES.

1. Find the equation whose roots are 1, 2, and 3.

Subtracting each root from x, we have the binomial factors,

$$x - 1, \quad x - 2, \quad \text{and} \quad x - 3;$$

multiplying these together, and placing their product equal to 0, we find,

$$x^3 - 6x^2 + 11x - 6 = 0,$$

which is the required equation.

2. Find the equation whose roots are -7 and -4.
Ans. $x^2 + 11x + 28 = 0$.

3. Find the equation whose roots are 3, 4, and -7.
Ans. $x^3 - 37x + 84 = 0$.

4. Find the equation whose roots are -1, -4, and -8. *Ans.* $x^3 + 13x^2 + 44x + 32 = 0$.

5. Find the equation whose roots are -2, -2, $+4$, and -4. *Ans.* $x^4 + 4x^3 - 12x^2 - 64x - 64 = 0$.

6. Find the equation whose roots are equal to -3, -3, and -3. *Ans.* $x^3 + 9x^2 + 27x + 27 = 0$.

7. Find the equation whose roots are 2, 3, 5, and -6. *Ans.* $x^4 - 4x^3 - 29x^2 + 156x - 180 = 0$.

8. Find the equation whose roots are 1, 2, and -5.
Ans. $x^3 + 2x^2 - 13x + 10 = 0$.

9. Find the equation whose roots are 3, 4, -1, and -6. *Ans.* $x^4 - 31x^2 + 42x + 72 = 0$.

10. Find the equation whose roots are -3, -4, $2 + 3\sqrt{-1}$, and $2 - 3\sqrt{-1}$.
Ans. $x^4 + 3x^3 - 3x^2 + 43x + 156 = 0$.

Simplification of the preceding Rule.

197. The results in the preceding article were found by actual multiplication; they might have been found by means of two simple laws. To demonstrate these laws, let a, b, c, . . . k, l, denote the roots of an equation of the form (1); then will its first member be equal to

$$(x-a)\,(x-b)\,(x-c)\;.\;.\;.\;(x-k)\,(x-l).$$

If we perform the multiplication as far as three factors, we have the result shown below:

$$\begin{array}{l}
x-a\\
x-b\\
\hline
x^2-(a+b)x+ab\\
x-c\\
\hline
\begin{array}{r|r|l}
x^3-a & x^2+ac & x-abc\\
-b & +bc & \\
-c & +ab &
\end{array}
\end{array}$$

Examining this product of three factors, we see that it is subject to the following laws of formation:

1°. *The exponent of x, in the first term, is equal to the number of factors, and it goes on diminishing by* 1, *in each term, to the right, to the last term, where it is* 0.

2°. *The coefficient of the first term is* 1; *that of the second term is equal to the algebraic sum of the roots with their signs changed; that of the third term is* *equal to the algebraic sum of the different products of the roots with their signs changed, taken in sets of* 2; *and the coefficient of the last term is equal to the continued product of all the roots with their signs changed.*

By a process analogous to that used in deducing the binomial formula it may be shown that these laws hold good for finding the product of any number of binomial factors of the kind considered; hence, we have the following rule for forming an equation whose roots are given:

RULE.

I. The exponent of the unknown quantity in the first term is equal to the number of roots, and goes on diminishing by 1, *in each term, to the right, to the last term, where it is* 0.

II. The coefficient of the first term is 1. *To find the remaining coefficients, change the signs of all the roots; find the algebraic sum of the results, and it will be the coefficient of the second term; find the algebraic sum of the different products of the results taken two in a set, and it will be the coefficient of the third term; find the algebraic sum of the different products of the results taken three in a set, and it will be the coefficient of the fourth term; proceed in this way, to the last term, which will be equal to the continued product of all the results.*

The last term, that is, the term that contains the 0 power of the unknown quantity, is called the **absolute term.**

If the sum of the positive roots is numerically equal to the sum of the negative roots, their algebraic sum will be 0, and consequently the coefficient of the second term of the equation will be 0, and that term will disappear from the equation; conversely, if the second term is wanting, the sum of the positive roots is equal to the sum of the negative roots.

EXAMPLES.

1. Find the equation whose roots are 1, 5, and 9.

The coefficient of x^3, is 1; the coefficient of x^2, is $-1-5-9 = -15$; the coefficient of x, is $(-1 \times -5) + (-1 \times -9) + (-5 \times -9) = +59$, and the absolute term is, $-1 \times -5 \times -9 = -45$; hence, the required equation is,

$$x^3 - 15x^2 + 59x - 45 = 0.$$

2. Find the equation whose roots are -3, -4, -5, and -6.

The coefficient of x^4, is 1; the coefficient of x^3, is $3+4+5+6 = 18$; the coefficient of x^2, is $3\times4+3\times5+3\times6+4\times5+4\times6+5\times6 = 119$; the coefficient of x, is $3\times4\times5+3\times4\times6+3\times5\times6+4\times5\times6 = 342$; and the absolute term is, $3\times4\times5\times6 = 360$; hence, the required equation is,

$$x^4 + 18x^3 + 119x^2 + 342x + 360 = 0.$$

3. Find the equation whose roots are -1, -2, and 3. *Ans.* $x^3 - 7x - 6 = 0.$

4. Find the equation whose roots are $-\frac{1}{2}$, $-\frac{1}{4}$, and $-\frac{1}{8}$.

Ans. $x^3 + \frac{7}{8}x^2 + \frac{7}{32}x + \frac{1}{64} = 0.$

5. Find the equation whose roots are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{5}$.

Ans. $x^3 - \frac{31}{30}x^2 + \frac{1}{3}x - \frac{1}{30} = 0.$

First Transformation.

198. *Every equation of the form* (1), *whose coefficients are fractional, can be reduced to another equation of the same form, whose coefficients are entire.*

Take the equation,

$$x^3 + \frac{7}{8}x^2 + \frac{7}{32}x + \frac{1}{64} = 0.$$

Substituting for x, the fraction $\frac{y}{k}$, in which y is a new unknown quantity, and k an arbitrary quantity; we have,

$$\frac{y^3}{k^3} + \frac{7}{8}\cdot\frac{y^2}{k^2} + \frac{7}{32}\cdot\frac{y}{k} + \frac{1}{64} = 0;$$

multiplying both members by k^3, we have,

$$y^3 + \frac{7k}{8}y^2 + \frac{7k^2}{32}y + \frac{k^3}{64} = 0. \quad . \quad (a)$$

Since k is arbitrary, we give it such a value as will make the coefficients, $\frac{7k}{8}$, $\frac{7k^2}{32}$, and $\frac{k^3}{64}$, whole numbers. This can always be done; because, if we make k equal to the least common multiple of the denominators, the different powers of k will be divisible by each denominator separately, and consequently, the coefficients will be whole numbers. Making $k = 64$, equation (a) becomes,

$$y^3 + 56y^2 + 896y + 4096 = 0.$$

It often happens, that a less value of k will make the coefficients entire, as in the above example. Resolving the denominators into factors, the coefficients become,

$$\frac{7k}{2.2.2}, \quad \frac{7k^2}{2.2.2.2.2}, \quad \frac{k^3}{2.2.2.2.2.2}.$$

We see that $k = 2 . 2 . 2$, will render all of these entire, giving,

$$7, \quad 14, \quad \text{and} \quad 8.$$

Hence, the transformed equation is,

$$y^3 + 7y^2 + 14y + 8 = 0.$$

If the roots of the last equation are known, those of the given equation may be found, by dividing each by 8, since $x = \frac{y}{8}$.

EXAMPLES.

1. Transform the equation, $x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0$, into one whose coefficients are entire, that of the first term being 1.

Making $x = \frac{y}{k}$, and multiplying both members by k^3,

$$y^3 - \frac{7k}{3}y^2 + \frac{11k^2}{36}y - \frac{25k^3}{72} = 0.$$

The fractional coefficients, with their denominators factored, are,

$$-\frac{7k}{3}, \quad \frac{11k^2}{2.2.3.3}, \quad \text{and} \quad -\frac{25k^3}{2.2.2.3.3};$$

making $k = 2 \times 3 = 6$, these coefficients become,

$$-14, \quad 11, \quad \text{and} \quad -75;$$

hence, the transformed equation is,

$$y^3 - 14y^2 + 11y - 75 = 0.$$

Any root of this equation, divided by 6, gives the corresponding root of the assumed equation.

2. Transform the equation $x^3 - \frac{41}{30}x^2 + \frac{1}{3}x - \frac{1}{30} = 0$, into one whose coefficients are entire.

Ans. $y^3 - 41y^2 + 300y - 900 = 0.$

3. Transform the equation, $x^3 - \frac{2}{25}x^2 + \frac{1}{30}x - \frac{1}{40} = 0$, into one having entire coefficients.

Ans. $y^3 - 12y^2 + 750y - 84375 = 0.$

4. Transform the equation, $x^3 + \frac{1}{4}x^2 + \frac{1}{16}x + \frac{1}{64} = 0$, into one whose coefficients are whole numbers.

Ans. $y^3 + y^2 + y + 1 = 0.$

Second Transformation.

199. *An equation of the form* (1), *may be transformed into another of the same form, in which the roots are any multiple of those of equation* (1).

For, substituting in (1), $\frac{y}{k}$ for x, and multiplying through by k^n, we have,

$$y^n + kpy^{n-1} + k^2qy^{n-2} + \ \ldots \ + k^{n-1}ty + k^nu = 0; \quad (9)$$

Equation (9) is of the same form as (1), and since $y = kx$, each root of (9) is equal to k times the corresponding root of equation (1); k may be entire or fractional.

EXAMPLES.

1. Transform the equation, $x^2 - 7x + 12 = 0$, into another in which the roots are twice as great.

If we make $x = \frac{y}{2}$, the resulting equation is,

$$\frac{y^2}{4} - \frac{7y}{2} + 12 = 0; \text{ or, } y^2 - 14 \;\; + 48 = 0.$$

2. Transform the equation, $x^2 - 4x - 21 = 0$, into another whose roots are three times as great.

Ans. $y^2 - 12y - 189 = 0.$

3. Transform the equation, $x^2 - 4x - 32 = 0$, into one whose roots are half as great.

Ans. $y^2 - 2y - 8 = 0.$

4. Transform the equation, $x^2 + 11x + 28 = 0$, into one whose roots are twice as great.

Ans. $y^2 + 22y + 112 = 0.$

5. Transform the equation, $x^4 + 4x^3 - 12x^2 - 64x - 64 = 0$, into one whose roots are half as great.

Ans. $y^4 + 2y^3 - 3y^2 - 8y - 4 = 0.$

Third Transformation.

200. *An equation of the form* (1), *may be reduced to another of the same form, whose roots differ from those of the given equation by any given quantity.*

For, let us make, $x = y + r$, in (1), y being a new unknown quantity, and r being arbitrary; the resulting equation will be,

$$(y + r)^n + p(y + r)^{n-1} + \ . \ . \ . \ t(y + r) + u = 0.$$

Developing the different powers of $y + r$, by the binomial formula, arranging the results according to the descending powers of y, and denoting the coefficients of the $n-1$, $n-2$, &c., powers of y, by p', q', &c., we have,

$$y^n + p'y^{n-1} + q'y^{n-2} + \ldots + s'y^2 + t'y + u' = 0, \quad . \ (a)$$

an equation of the proposed form, whose roots are less than those of (1) by r, since $y = x - r$, by hypothesis.

The operation of making this transformation is somewhat tedious, but a simple rule may be deduced for finding the coefficients of the transformed equation, that will render the transformation sufficiently expeditious.

Equation (a) was derived from (1), by making $x = y + r$; hence, if we make $y = x - r$, in equation (a), the resulting equation will be identical with equation (1); making this substitution, we have,

$$(x-r)^n + p'(x-r)^{n-1} + \ldots + s'(x-r)^2 + t'(x-r) + u' = 0. \quad . \quad . \quad (b)$$

Now, if the first member of (b), or what is the same thing, the first member of equation (1), is divided by $x - r$, the remainder will be equal to u', the coefficient of y^0, in equation (a): the quotient obtained is,

$$(x-r)^{n-1} + p'(x-r)^{n-2} \ldots . + s'(x-r) + t'.$$

If this quotient is divided by $x - r$, the remainder will be t', the coefficient of y, in equation (a). If the second quotient be divided by $x - r$, the remainder will be the coefficient of y^2 in equation (a); and so on. Continuing this process of division, the successive remainders are the coefficients of the transformed equation, taken in a reverse order.

EXAMPLES.

1. Find an equation whose roots are less by 2, than those of the equation, $x^3 - 6x^2 - 7x + 60 = 0$.

OPERATION.

$$\begin{array}{r|l|l|l}
x^3 - 6x^2 - 7x + 60 & \underline{x - 2} & & \\
\underline{x^3 - 2x^2} & x^2 - 4x - 15 & \underline{x - 2} & \\
- 4x^2 - 7x & \underline{x^2 - 2x} & x - 2 & \underline{x - 2} \\
\underline{- 4x^2 + 8x} & - 2x - 15 & \underline{x - 2} & 1 \\
-15x + 60 & \underline{- 2x + 4} & 0 & \\
\underline{-15x + 30} & - 19 & & \\
+ 30 & & &
\end{array}$$

Here, we divide $x^3 - 6x^2 - 7x + 60$, by $x - 2$, and find a quotient, $x^2 - 4x - 15$, with a remainder, $+ 30$; we next divide $x^2 - 4x - 15$, by $x - 2$, and find a quotient, $x - 2$, with a remainder, $- 19$; we next divide $x - 2$, by $x - 2$, and find a quotient, 1, with a remainder, 0: hence, the transformed equation is,

$$y^3 - 19y + 30 = 0.$$

The roots of the given equation are, 4, 5, and -3, and those of the transformed equation are, 2, 3, and -5, as may be shown by forming the equations in the two cases.

2. Find an equation whose roots are greater by 1, than those of the equation, $x^3 + 4x^2 + 5x + 7 = 0$.

Ans. $y^3 + y^2 + 5 = 0$.

3. Find an equation whose roots are less by 1, than those of the equation, $x^3 - 7x + 7 = 0$.

Ans. $y^3 + 3y^2 - 4y + 1 = 0$.

4. Find an equation whose roots are greater by 10, than those of the equation, $x^4 + 42x^3 + 663x^2 + 4664x = 0$.

Ans. $y^4 + 2y^3 + 3y^2 + 4y - 12340 = 0$.

Synthetic Division.

201. The operation of successive division may be abridged by the method of synthetic division, which will now be explained, as far as necessary for our purpose.

If a polynomial of the form,

$$x^n + px^{n-1} + qx^{n-2} + \ldots + sx^2 + tx + u,$$

is divided by a binomial of the form, $x - r$, or $x + r$, the quotient will be of the form,

$$x^{n-1} + p'x^{n-2} + \&c. + s'x + t',$$

and the remainder will be independent of x.

Now, as our object is to find the remainder, we need only consider the coefficients in the dividend, divisor, and quotient; to illustrate, let us take example 1, Art. 200: writing down the coefficients only, we have,

OPERATION.

$$\begin{array}{r|r|r|l}
1-6-7+60 & 1-2 & & \\ \cline{2-2}
1-2 & 1-4-15 & 1-2 & \\ \cline{1-1}\cline{3-3}
-4-7 & 1-2 & 1-2 & 1-2 \\ \cline{2-2}\cline{4-4}
-4+8 & -2-15 & 1-2 & 1 \\ \cline{1-1}\cline{3-3}
-15+60 & -2+4 & 0 & \\ \cline{2-2}
-15+30 & -19 & & \\ \cline{1-1}
+30 & & &
\end{array}$$

Proceeding as in the previous article, neglecting the different powers of x in each term, we find the first remainder equal to 30, the second remainder equal to -19, and the third remainder equal to 0.

But this operation may be abbreviated; for, if we examine the process we see that the first term of the quotient in each division is 1, the second term of the quotient is obtained by multiplying the first term by the second term of the divisor, and subtracting the result from the second term of the dividend, or, what is the same thing, multiplying the first term of the quotient, by the second term of the divisor with its sign changed, and adding the result to the second term of the dividend; the third term of the quotient is found by multiplying the second term, by the second term of the divisor, with its sign changed, and adding the result to the third term of the dividend: and so on, till the last term of the dividend has been used: the last result is the remainder required.

The above process, when written out, takes the form,

$$\begin{array}{lll}
 & 1 - 6 - 7 + 60 \;|\; \underline{+\,2} & \\
 & \underline{+ 2 - 8 - 30} & \\
\textit{1st quotient,} & 1 - 4 - 15, + 30 & \textit{1st remainder.} \\
 & \underline{+ 2 - 4} & \\
\textit{2d quotient,} & 1 - 2, - 19 & \textit{2d remainder.} \\
 & \underline{+ 2} & \\
\textit{3d quotient,} & 1, + 0 & \textit{3d remainder.}
\end{array}$$

Here, we have changed the sign of the second term of the divisor, and dropped the first term entirely; the first term of the first quotient is 1; multiplying 1 by 2, we find 2, which we write under -6, and adding -6 and $+2$, we find -4, which we write for the second term of the quotient; multiplying -4 by 2, we find -8, which we write under -7, and adding -7 and -8, we get -15, which we write for the third term of the quotient; multiplying -15 by 2, we get -30, which we place under 60, and adding, we get 30 for the first remainder, which we point off by a comma.

Commencing the second division, we write 1 for the first term of the quotient; multiplying by 2, and adding to -4, we get -2, for the second term; multiplying this by 2, and adding to -15, we get -19 for the second remainder.

Commencing the third division, we write 1 for the first term of the quotient; multiplying by 2, and adding to -2, we get 0 for the third remainder; hence, the required equation is,

$$y^3 - 19y + 30 = 0.$$

In the same manner, all similar transformations may be effected, as is shown in the following examples:

5. Find an equation whose roots are greater by 3 than those of the equation,

$$x^4 - 17x^3 + 12x^2 - 33x + 67 = 0.$$

Here, the divisor is $x + 3$, and the second term, with its sign changed, is -3.

OPERATION.

$$\begin{array}{rrrrrl}
1 & -17 & +12 & -33 & +67 & \underline{|\,-3.} \\
 & -3 & +60 & -216 & +747 & \\
\hline
1 & -20 & +72 & -249, & +814 & \ldots\ \textit{1st remainder.} \\
 & -3 & +69 & -423 & & \\
\hline
1 & -23 & +141, & -672 & & \ldots\ldots\ \textit{2d remainder.} \\
 & -3 & +78 & & & \\
\hline
1 & -26, & +219 & & & \ldots\ldots\ldots\ \textit{3d remainder.} \\
 & -3 & & & & \\
\hline
1, & -29 & & & & \ldots\ldots\ldots\ldots\ \textit{4th remainder.}
\end{array}$$

Hence, the required equation is,

$$y^4 - 29y^3 + 219y^2 - 672y + 814 = 0.$$

It is not necessary that the coefficient of the first term of the dividend should be 1. It may be any number, and in that case the first term of each quotient will be the same number.

6. Find the equation whose roots are less by 3, than those of the equation,

$$3x^4 - 4x^3 + 7x^2 + 8x - 12 = 0.$$

OPERATION.

$$\begin{array}{rrrrrl}
3 & -4 & +7 & +8 & -12 & \underline{|\,+3} \\
 & +9 & +15 & +66 & +222 & \\
\hline
3 & +5 & +22 & +74, & +210 & \ldots\ \textit{1st remainder.} \\
 & +9 & +42 & +192 & & \\
\hline
3 & +14 & +64, & +266 & & \ldots\ldots\ \textit{2d remainder.} \\
 & +9 & +69 & & & \\
\hline
3 & +23, & +133 & & & \ldots\ldots\ldots\ \textit{3d remainder.} \\
 & +9 & & & & \\
\hline
3, & +32 & & & & \ldots\ldots\ldots\ldots\ \textit{4th remainder.}
\end{array}$$

Hence, the required equation is,

$$3y^4 + 32y^3 + 133y^2 + 266y + 210 = 0.$$

If any terms are wanting in the dividend, their place must be supplied by 0's.

7. Find an equation whose roots are greater by 3, than those of the equation, $x^4 - 2 = 0$.

Ans. $y^4 - 12y^3 + 54y^2 - 108y + 79 = 0$.

8. Find an equation whose roots are less by 0.1, than those of the equation, $x^3 - 1 = 0$.

Ans. $y^3 + 0.3y^2 + 0.03y - 0.999 = 0$.

9. Find an equation whose roots are greater by 3, than those of the equation, $x^4 + 13x^3 + x^2 - 11 = 0$.

Ans. $y^4 + y^3 - 62y^2 + 237y - 272 = 0$.

10. Find an equation whose roots are less by 2, than those of the equation, $x^4 - 9x^3 + 20x^2 - 10x - 1 = 0$.

Ans. $y^4 - y^3 - 10y^2 - 6y + 3 = 0$.

If the second term of the divisor used is equal to the quotient of the coefficient of the second term of the dividend with its sign changed, by the exponent denoting the degree of the dividend, the last remainder will, from the nature of the case, be equal to 0, and the second term of the resulting equation will be 0, or *wanting*.

11. Transform the equation, $x^3 - 6x^2 + 7x - 10 = 0$, so that the resulting equation shall want the second term.

OPERATION.

$$\begin{array}{rrrr|l}
1 & -6 & +7 & -10 & \underline{2} \\
 & +2 & -8 & -2 & \\ \cline{1-4}
1 & -4, & -1, & -12 & \ldots\ldots\ldots\ \textit{1st remainder.} \\
 & +2 & -4 & & \\ \cline{1-3}
1 & -2, & -5 & & \ldots\ldots\ldots\ldots\ \textit{2d remainder.} \\
 & +2 & & & \\ \cline{1-2}
1, & +0 & & & \ldots\ldots\ldots\ldots\ldots\ \textit{3d remainder.}
\end{array}$$

Hence, the required equation is,

$$y^3 - 5y - 12 = 0.$$

The roots of the resulting equation are less by 2, than those of the given equation.

Transform the following equations, so that the resulting equations shall want their second terms:

12. $x^3 + 9x^2 - x + 4 = 0$. *Ans.* $y^3 - 28y + 61 = 0$.

The roots of the resulting equation are greater by 3, than those of the given equation.

13. $x^4 - 8x^3 + 7x^2 + 3x + 4 = 0$.
Ans. $y^4 - 17y^2 - 33y - 10 = 0$.

Fourth Transformation.

202. *An equation of the form* (1), *may be transformed into another of the same form whose roots are equal those of the given equation, with their signs changed.*

If in (1), we make $x = -y$, we shall have the equation,

$$(-y)^n + p(-y)^{n-1} + \&c. + t(-y) + u = 0; \quad (10)$$

which is of the proposed form.

The following examples show what changes will result in making the proposed transformation:

First, if the degree of the equation is even, as,

$$x^4 + 2x^3 - 7x^2 + x - 8 = 0;$$

making $x = -y$, we have,

$$y^4 - 2y^3 - 7y^2 - y - 8 = 0.$$

Here, the coefficients of the terms remain numerically the same as before, but the signs of the coefficients of the odd powers are changed. The same will hold true in all equations of an even degree, when thus transformed.

Second, if the degree of the equation is odd, as,

$$x^5 - 7x^4 + x^3 - 2x^2 - x - 1 = 0;$$

making $x = -y$, and dividing both members by -1, we have,

$$y^5 + 7y^4 + y^3 + 2y^2 - y + 1 = 0.$$

Here, the numerical values of the coefficients remain the same, but the signs of the coefficients of the even powers are changed. The same will hold true in all equations of an odd degree, when thus transformed.

In both cases, the transformation may be made by changing the signs of the second, fourth, sixth, &c., terms. We have supposed the equation complete; when it is incomplete, the wanting terms must be supplied by 0's, in making the transformation by the last rule.

Fifth Transformation.

203. *An equation of the form* (1), *may be transformed into another of the same form, whose roots are equal to the reciprocals of those of the given equation.*

If we substitute $\frac{1}{y}$ for x, in equation (1), we have,

$$\frac{1}{y^n} + \frac{p}{y^{n-1}} + \frac{q}{y^{n-2}} + \ldots + \frac{s}{y^2} + \frac{t}{y} + u = 0;$$

multiplying both members of this equation, by $\frac{y^n}{u}$, and writing the terms in an inverse order, we have,

$$y^n + \frac{t}{u}y^{n-1} + \frac{s}{u}y^{n-2} + \ldots + \frac{q}{u}y^2 + \frac{p}{u}y + \frac{1}{u} = 0; \quad (11)$$

which is of the required form. The coefficients are found by writing those of the given equation in an inverse order, and dividing each by the absolute term. If any term is wanting in the given equation, the term at the same distance from the other extreme will be wanting in the required equation.

EXAMPLES.

Transform the following equations into others whose roots shall be the reciprocals of those of the given equations:

1. $x^3 + 3x^2 + 9x + 3 = 0.$

Ans. $y^3 + 3y^2 + y + \frac{1}{3} = 0.$

2. $4x^4 + 6x^3 + 2x + 2 = 0.$

Ans. $y^4 + y^3 + 3y + 2 = 0.$

3. $7x^2 - 7x - 2 = 0.$ *Ans.* $y^2 + \frac{7}{2}y - \frac{7}{2} = 0.$

II. DERIVED EQUATIONS AND EQUAL ROOTS.

Definitions.

204. A **derived polynomial**, is one that may be derived from a given polynomial, by multiplying each term by the exponent of the leading letter in that term, and then diminishing the exponent of the leading letter, by 1, in each result; the derived polynomial is also called the **derivative**; and that from which it is derived, is called the **primitive**. Thus, $x^3 + 2x^2 + 3x + 1$, being a given *primitive* polynomial, its derivative is, $3x^2 + 4x + 3$.

A **derived equation**, or a *derivative equation*, is one whose members are derivatives of the two members of a given equation. Thus, $3x^2 + 2x + 5 = 0$, being a primitive equation, its *derivative* is, $6x + 2 = 0$.

EXAMPLES.

Find the derivatives of the following equations:

1. $x^4 + 3x^2 + 2 = 0$. *Ans.* $4x^3 + 6x = 0$.

2. $x^4 + 6x^3 + 20x^2 + 10 = 0$.
Ans. $4x^3 + 18x^2 + 40x = 0$.

The derivative of the equation,

$$x^n + px^{n-1} + qx^{n-2} + \ldots + sx^2 + tx + u = 0, \quad . \quad (1)$$

is,

$$nx^{n-1} + (n-1)px^{n-2} + (n-2)qx^{n-3} + \ldots + 2sx + t = 0. \quad . \quad (12).$$

Equal Roots.

205. We have seen that the first member of equation (1) is composed of as many binomial factors of the form, $x - a$, $x - b$, &c., as there are roots, each of which corresponds to a root of the equation. When the equation has two roots equal to a, there will be two factors equal to $x - a$, that is, the first member will be divisible by $(x - a)^2$; when there are three roots equal to a, the first member will be divisible by $(x - a)^3$, and so on. To deduce a general rule for determining whether an equation has equal roots, and for freeing it of them, let us resume the general equation,

$$x^n + px^{n-1} + qx^{n-2} + \ldots + sx^2 + tx + u = 0 ; \ . \ (1)$$

substituting $y + r$ for x, we have,

$$(y + r)^n + p(y + r)^{n-1} + q(y + r)^{n-2} + \ldots + s(y + r)^2$$
$$+ t(y + r) + u = 0 ;$$

developing the different powers of $y + r$, by the binomial formula, and arranging the result according to the ascending powers of y, we have,

$$\left.\begin{array}{l} r^n \\ + pr^{n-1} \\ + qr^{n-2} \\ \vdots \\ + sr^2 \\ + tr \\ + u \end{array}\right| y^0 + \left.\begin{array}{l} nr^{n-1} \\ + (n-1)pr^{n-2} \\ + (n-2)qr^{n-3} \\ \vdots \\ + 2sr \\ + t \\ \end{array}\right| y + \ldots + y^n = 0 \ . \ . \ (a);$$

in this equation, the values of y are equal to the values of x, in equation (1), each diminished by r, since, by hypothesis, we have,

$$x = y + r ; \quad \text{whence,} \quad y = x - r.$$

Now, r is entirely arbitrary; we may therefore assign a value to it at pleasure. If we suppose r to be a root of equation (1), the coefficient of y^0, in equation (a), will be equal to 0, because that coefficient is what the first member of (1) becomes, when r is substituted for x (Art. 193); making this coefficient equal to 0, and dividing each of the remaining terms by y, we have,

$$\left.\begin{array}{l} \quad nr^{n-1} \\ +\,(n-1)pr^{n-2} \\ +\,(n-2)qr^{n-3} \\ \qquad \vdots \\ +\ 2sr \\ +\ t \end{array}\right| \, y^0 + \ . \ . \ . \ . + y^{n-1} = 0 \ . \ . \quad (b).$$

Now, the $n-1$ roots of equation (b), are equal to the different values of $x-r$, in equation (1), that is, to the results obtained by subtracting the root r of equation (1), from each of the other roots in succession. If, therefore, equation (1) has two roots equal to r, one of these differences will be equal to 0, that is, one of the roots of equation (b), will also be equal to 0. But, when $x-r$ is equal to 0, that is, when y is equal to 0, all of the terms of equation (b), except the first, reduce to 0, consequently that term must be separately equal to 0; or,

$$nr^{n-1} + (n-1)pr^{n-2} + (n-2)qr^{n-3} + \\ \ldots + 2sr + t = 0 \ . \ . \ . \ . \ . \quad (c).$$

Comparing this equation with equation (12), (Art. 204), we see that it is what the derivative of equation (1) becomes, when for x we substitute r. Hence, we conclude that when equation (1) has two roots equal to r, its derivative has 1 root equal to r; and conversely, when the derivative has 1 root equal to r, the primitive has 2 roots equal to r.

By a similar course of reasoning, it may be shown that, when equation (1) has 3, 4, 5, &c., roots, each equal to r, its derivative has 2, 3, 4, &c., roots, each equal to r; and conversely, when the derivative has 2, 3, 4, &c., roots, each equal to r, the primitive has 3, 4, 5, &c., roots, each equal to r; that is, when the first member of the given equation is divisible by $(x-r)^2$, $(x-r)^3$, &c., the first member of its derivative will be divisible by $x-r$, $(x-r)^2$, &c., and the reverse; hence, we have the following rule for determining whether an equation has any equal roots; and for freeing the equation from them if it has any:

RULE.

Find the derivative equation; then apply the rule for the greatest common divisor to the first members of the primitive and derivative equations; if no common divisor is found, the equation has no equal roots; if a common divisor is found, divide both members of the given equation by it, and the resulting equation will have no equal roots.

The operation of finding the values of equal roots, consists in placing the greatest common divisor found equal to 0, and solving the resulting equation. When this can be done, the equal roots may all be found. For each single root of the resulting equation, there will be two equal roots in the primitive equation; for each pair of equal roots in the resulting equation, there will be three equal roots in the given equation; and so on, as indicated in the preceding discussion. The equation found, by placing the greatest common divisor equal to 0, is to be treated, in all respects, like an original equation; and of course, the process for equal roots may be applied to it; and so on, indefinitely.

EXAMPLES.

1. Eliminate the equal roots from the equation,

$$x^5 - 27x^3 + 22x^2 + 192x - 288 = 0.$$

The derivative equation is,

$$5x^4 - 81x^2 + 44x + 192 = 0.$$

The greatest common divisor of the first members of the given equation and its derivative, is, $(x - 3)(x + 4)$; dividing both members of the given equation by this, we have the equation,

$$x^3 - x^2 - 14x + 24 = 0,$$

which has no equal roots.

If we place the common divisor equal to 0, we have,

$$(x - 3)(x + 4) = 0. \qquad \therefore \quad x = 3, \text{ and } \quad x = -4.$$

Hence, the given equation has two roots equal to 3, and two roots equal to -4. Dividing both members of the given equation, by $(x - 3)^2 (x + 4)^2$, there results,

$$x - 2 = 0; \qquad \therefore \quad x = 2.$$

The given equation is completely solved, and, in like manner, many other equations may be treated.

2. Eliminate the equal roots from the equation,

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = 0.$$

The derivative equation is,

$$4x^3 - 15x^2 + 18x - 7 = 0;$$

the greatest common divisor of the first member is,

$$(x - 1)^2 ;$$

hence, the required equation is,

$$x^2 - 3x + 2 = 0 ;$$

which may be solved by known rules. The roots of the given equation are, 1, 1, 1, and 2.

III. SOLUTION OF HIGHER EQUATIONS.

206. If an equation of the form (1) has equal roots, it may be freed from them by the preceding principle, and the resulting equation will be of a lower degree than the primitive one; which is always to be desired. We shall, in what follows, suppose that the equation in question has been freed of its equal roots.

It was shown, in Art. 197, that the absolute term of an equation of the form (1), is equal to the continued product of the roots with their signs changed. When this absolute term is a whole number, and it may always be made so (Art. 198), and any root is a whole number, it may often be found by trial; for, we may resolve the absolute term into its factors, and then, by the process of synthetic division, we see whether the first member of the given equation is exactly divisible by the unknown quantity, increased by any one of these factors; if so, that factor, with its sign changed, is a root (Art. 193); if the first member is not divisible by the unknown quantity, increased by any one of the factors, then the equation has no entire roots.

EXAMPLES.

1. Find the entire root in the equation,

$$x^3 + 3x^2 + 9x - 38 = 0.$$

The divisors of -38 are, ± 1, ± 2, ± 19, ± 38.

We see, at a glance, that neither $+1$, nor -1, will satisfy the equation; hence, neither -1 or $+1$, can be a root.

By applying the rule for synthetic division we see that the first member of the given equation is divisible by $x - 2$; hence, $+2$ is a root. Dividing both members of the given equation by $x - 2$, we have,

$$x^2 + 5x + 19 = 0;$$

which can be solved by known rules. Both of its roots are imaginary, one equal to $\frac{-5 + \sqrt{-51}}{2}$, and $\frac{-5 - \sqrt{-51}}{2}$.

2. Find the entire root of the equation,

$$x^3 - 12x^2 + 4x + 207 = 0.$$

The divisors of 207 are,

$$\pm 1, \ \pm 3, \ \pm 9, \ \pm 23, \ \pm 69, \text{ and } \pm 207;$$

testing each factor, we find that the first member of the given equation is divisible by $x - 9$; hence, 9 is a root.

When freed of this root, the given equation becomes,

$$x^2 - 3x - 23 = 0,$$

which gives the roots, $\frac{3 + \sqrt{101}}{2}$, and $\frac{3 - \sqrt{101}}{2}$.

In the same way, the following equations may be solved:

3. $x^3 + 3x^2 - 6x - 8 = 0.$

Ans. $x' = 2$, $x'' = -1$, and $x''' = -4$.

4. $x^3 + 9x - 1430 = 0$.

$$x' = 11,\ x'' = \frac{-11 + \sqrt{-399}}{2},\ \text{and } x''' = \frac{-11 - \sqrt{-399}}{2}.$$

5. $x^3 - 1 = 0$.

$$x' = 1,\ x'' = \frac{-1 + \sqrt{-3}}{2},\ \text{and } x''' = \frac{-1 - \sqrt{-3}}{2}.$$

Nature of the Roots of Equation (1).

207. An equation of the form (1), whose coefficients are entire, has no rational roots that are irreducible fractions. For, if so, suppose one of these roots to be of the form $\frac{a}{b}$, an irreducible fraction.

Substituting this fraction for x, in equation (1), and transposing all of the terms, except the first, to the second member, we have,

$$\frac{a^n}{b^n} = -p\frac{a^{n-1}}{b^{n-1}} - q\frac{a^{n-2}}{b^{n-2}} - \ldots - t\frac{a}{b} - u;$$

multiplying both members by b^{n-1}, we have,

$$a^{n-1} \times \frac{a}{b} = -pa^{n-1} - qa^{n-2}b - \ldots - tab^{n-2} - ub^{n-1};$$

the first member of this equation is an irreducible fraction, and the second member is entire: which is manifestly impossible; hence, the supposition that a root can be an irreducible fraction, is absurd.

Imaginary Roots.

208. If an equation of the form (1), whose coefficients are real, has any imaginary roots, each must be of the form $a + b\sqrt{-1}$, or $a - b\sqrt{-1}$, (Art. 134).

It may be shown that imaginary roots always enter by pairs, that is, if $a + b\sqrt{-1}$, is a root, then will $a - b\sqrt{-1}$, be a root, also; for, substituting $a + b\sqrt{-1}$ for x, in equation (1), we have,

$$(a + b\sqrt{-1})^n + p(a + b\sqrt{-1})^{n-1} + \\ \dots + t(a + b\sqrt{-1}) + u = 0; \quad . \quad . \quad . \quad (a)$$

if we develop the different powers of $a + b\sqrt{-1}$, by the binomial formula, and perform the operations indicated, the first member of the resulting equation will be made up of two kinds of terms, *real* and *imaginary*. The imaginary terms arise from the odd powers of $b\sqrt{-1}$, and consequently (Art. 138), they contain no other imaginary factor than $\sqrt{-1}$; hence, denoting the sum of the real quantities by m, and the sum of the coefficients of $\sqrt{-1}$, by n, the equation becomes,

$$m + n\sqrt{-1} = 0; \quad . \quad . \quad . \quad (b)$$

but, from principle 2°, Art. 138, equation (b) can only be satisfied, by making $m = 0$, and $n = 0$.

If we substitute $a - b\sqrt{-1}$, for x, in equation (1), and perform the indicated operations, the result will differ from that expressed by equation (b), only in the signs of the odd powers of $b\sqrt{-1}$. Hence, the resulting equation will be,

$$m - n\sqrt{-1} = 0; \quad . \quad . \quad . \quad (c)$$

but, we have shown that $m = 0$, and $n = 0$; hence, equation (c) is satisfied; and consequently, the substitution of $a - b\sqrt{-1}$ for x, in equation (1), satisfies it, which shows that $a - b\sqrt{-1}$ is a root.

If $a + b\sqrt{-1}$, is a root of equation (1), the first member of that equation must be divisible by $x - (a + b\sqrt{-1})$; and from what has just been proved, it must also be divisible by $x - (a - b\sqrt{-1})$; and consequently, it must be divisible by the product of these factors: but, the product of these two factors is positive for all real values of x (principle 3°, Art. 138); we therefore, conclude that *the number of imaginary roots is always even*, since they enter by pairs; and that *the product of all the binomial factors that correspond to imaginary roots is positive* for all real values of x.

The number of real roots, and consequently the number of imaginary roots, of an equation of the form (1), can be determined by a principle called **Sturm's theorem.**

Object of Sturm's Theorem.

209. If we denote the real roots of equation (1), by a, b, c, &c., and suppose them arranged according to their values, so that a shall be the least algebraically, that is, nearest to $-\infty$, b the next greater, and so on; and if we denote the product of all the binomial factors corresponding to imaginary roots, which is always positive, by Y, equation (1) may be written,

$$(x - a)(x - b)(x - c) \ldots Y = 0 \quad . \quad . \quad (13)$$

If we suppose $x = -\infty$, each of the factors $(x - a)$, $(x - b)$, &c., will be negative; and the first member will be negative, when there is an odd number of real roots, and positive when there is an even number; if we suppose x to increase from $-\infty$ towards $+\infty$, assuming in succession every possible value, the sign of the first member of (13), will remain unchanged, until x becomes equal to a, when the first member will become equal to 0; for all values of x, between a and b, the factor $x - a$ will be positive, and all the remaining ones will be negative; hence, the product of all the factors, or the first member of (13), will have a different sign from what it had before, that is, the first member will change its sign from $+$ to $-$, or from $-$ to $+$, when the value of x passes the real root a. When x becomes equal to b, the first member again becomes 0; for all values of x between b and c, the first two factors are positive, and all the remaining ones negative; hence, in this case, the sign of the first member is the same as in the first instance, that is, the sign of the first member changes from $-$ to $+$, or from $+$ to $-$, when the value of x passes the real root b. In the same way it may be shown, that the sign of the first member changes from $+$ to $-$, or from $-$ to $+$, whenever the value of x passes a real root, and that it does not change in any other case.

If, therefore, we suppose x to assume every possible value, from $-\infty$ up to $+\infty$, and determine the number of times that the first member changes sign, we shall have the number of real roots, and consequently the number of imaginary roots in the equation. The object of Sturm's Theorem is, to show the manner of determining the number of such changes of sign.

The demonstration of STURM'S THEOREM depends on the following principles:

Principles.

210. Let the first member of equation (1), after having been freed of its equal roots (Art. 205), be denoted by V, and let the derived polynomial of V be denoted by V_1. If we apply to V and V_1 the rule for finding their greatest common divisor, differing only in this respect, that instead of using the successive remainders as obtained, we *change their signs*, and also take care neither to introduce or reject any factors except positive ones, we shall ultimately arrive at a remainder that is independent of x; this remainder cannot be equal to 0, because V and V_1 have no common divisor, the equation $V = 0$ having been freed of its equal roots.

If we denote the several remainders, after their signs have been changed, by

$$V_2,\ V_3,\ \ldots\ V_n,\ V_{n+1},\ V_{n+2}\ \ldots\ V_{r-2},\ V_{r-1},\ V_r.$$

(which are read V *two*, V *three*, &c.), and if we denote the corresponding quotients by

$$Q_1,\ Q_2,\ \ldots\ Q_n,\ Q_{n+1},\ \ldots\ Q_{r-1},$$

the operation above indicated may be expressed as follows:

$$\frac{V}{V_1} = Q_1 - \frac{V_2}{V_1}$$

$$\frac{V_1}{V_2} = Q_2 - \frac{V_3}{V_2}$$

&c., &c.,

from which we have the following equations:

$$\begin{array}{lll} V & = V_1 \ Q_1 & - V_2 \\ V_1 & = V_2 \ Q_2 & - V_3 \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ V_{n-1} & = V_n \ Q_n & - V_{n+1} \\ V_n & = V_{n+1} \ Q_{n+1} & - V_{n+2} \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \\ V_{r-2} & = V_{r-1} \ Q_{r-1} & - V_r \end{array}$$

The quantities V_1, V_2, &c., are called *derivatives* of V. A value of x that reduces V or any of its derivatives to 0, is called a *root value* of the corresponding expression.

It is evident from the preceding equations that no two consecutive derivatives of V can have the same root value: for suppose the expressions V_{n-1} and V_n to reduce to 0 for the value $x = k$; then from the relation between these and V_{n+1} we should have the latter equal to 0 for the same value of x; from the next equation we should in like manner have V_{n+2} equal to 0; and so on to the last, in which we should have V_r equal to 0, which is impossible; hence, the following principle:

1°. *No two consecutive derivatives of* V *can reduce to* 0 *for the same value of x.*

If one of the derivatives of V reduces to 0 for any value of x, the preceding and following one will have contrary signs for that value. Thus, if $x = a$ reduces V_n to 0, we have from the relation between V_{n-1}, V_n, and V_{n+1}

$$V_{n-1} = - V_{n+1};$$

hence we have the following principle:

2°. *If any value of x reduces one of the derivatives of* V *to* 0, *the preceding derivative and the following derivative will have contrary signs for that value.*

If we denote what V becomes when we make x equal to $r + y$ by the symbol $V_{x=r+y}$, we shall have from equation (a), Art. 205,

$$V_{x=r+y} = A + A'y + A''y^2 + \&c.,$$

in which A denotes what V becomes when we make $x = r$, A′ is what V_1 becomes when we make $x = r$, &c.; now, if we suppose r to be a root of the equation $V = 0$, we shall have $A = 0$, and the preceding equation will become,

$$V_{x=r+y} = y(A' + A''y + \&c.)$$

It is plain that y may have a value so small that the sum of all the terms within the parenthesis after A′ will be numerically less than A′; for, if y is made inappreciably small in comparison with 1, then will $A''y$ be inappreciably small in comparison with A′; and since the following terms contain higher powers of y each will be inappreciably small with respect to the preceding one; and because the number of terms is finite, the sum of all after the first, must be less than the first; hence, when y is inappreciably small in comparison with 1, the sign of the sum of the terms within the parenthesis, must be the same as that of A′. Now if we make y inappreciably small with respect to 1, first negative and then positive, we shall have the equations,

$$V_{x=r-y} = -y(A' - A''y + \&c.),$$
$$V_{x=r+y} = +y(A' + A''y + \&c.);$$

in the second of these equations the sign of the second member is the same as the sign of A', and in the first it is the reverse; hence, the following principle:

3°. *If we make* x *inappreciably smaller than one of the real roots of the equation* $V = 0$, *the corresponding values of* V *and* V_1 *will have contrary signs; if we make* x *inappreciably larger than the same root, the corresponding values of* V *and* V_1 *will have the same sign.*

It was shown in Art. 209, that as x increases by inappreciably small increments from $-\infty$ to $+\infty$, the sign of V changes from $+$ to $-$, or from $-$ to $+$, whenever x passes one of the real roots of the equation $V = 0$. In like manner it may be shown that each of the derivatives of V changes sign whenever x passes a *root value* of that derivative; it is also obvious that no derivative of V can change sign under any other circumstances; hence, the following principle:

4°. *Each derivative of* V *changes sign whenever the value of* x *passes a root value of that derivative, and does not change sign under any other circumstances.*

Demonstration of Sturm's Theorem.

211. Let us write the quantities $V, V_1, \ldots V_n \ldots V_r$ in a column; then let us substitute the same value of

x in each, and write the signs of the results in a parallel column. In passing down the column of signs, whenever two consecutive ones are alike, there is said to be a **permanence**; and whenever two consecutive ones are unlike, there is said to be a **variation.**

If we now suppose x to increase by inappreciably small increments from $-\infty$ to $+\infty$, we shall find in succession values that will reduce V or some of its derivatives to 0, but from principle 1° no value of x will reduce two consecutive ones to 0. There may be two cases: *first*, when V reduces to 0, and *secondly*, when one of its derivatives reduces to 0.

First. Suppose that any value of x, as $x = k$, reduces V to 0; then is k a real root of the equation $V = 0$; it follows from principle 3° that for the value of x next preceding k, V and V_1 have contrary signs, and there is a *variation;* but for the value of x next following k, V and V_1 have the same signs, and there is a *permanence:* hence, every time that the value of x passes a real root of the equation $V = 0$, there is a variation lost, or converted into a permanence.

Secondly. Suppose that any value of x, as $x = m$, reduces one of the derivatives of V, as V_n, to 0; then is m a root value of V_n; it follows from principle 2° that V_{n-1} and V_{n+1} have contrary signs for this value of x; it follows from principle 4° that V_n changes sign when the value of x passes from the value immediately preceding m to the value immediately following m; it also follows from principles 1° and 4° that V_{n-1} and V_{n+1} do not change sign for these values of x.

Now, for the value of x immediately preceding m, V_{n-1} and V_{n+1}, have contrary signs, and V_n must have the same sign as one of them; there is therefore, one corresponding *variation,* and but one: for $x = m$ V_{n-1} and V_{n+1} have contrary signs, and V_n is 0; there is therefore, one corresponding *variation,* and but one: for the value of x next following m, V_{n-1} and V_{n+1} have contrary signs, and V_n must have the same sign as one of them; there is therefore, one corresponding *variation,* and but one: hence, whenever the value of x passes a root value of any derivative of V, there is no variation either lost or gained.

We therefore conclude that there is one variation lost in the column of signs whenever x passes a real root of the equation $V = 0$, and that there is no variation either lost or gained under any other circumstances; hence,

If we make $x = -\infty$ *in* V *and its derivatives, and write the corresponding signs in a column, and then if we make* $x = +\infty$ *in the same expressions and write the corresponding signs in a second column, the number of variations in the first column, diminished by the number of variations in the second column will be the number of real roots in the equation* $V = 0$.

This is STURM'S theorem.

When we make $x = -\infty$, or $x = +\infty$, in V, or in any of its derivatives, the value of the first term in each will be infinitely great with respect to all the following terms, and consequently, the sign of each result will be the same as the sign of its first term.

Illustration.

Let it be required to find the number of real roots in the equation,

$$x^4 - 12x^2 + 12x - 3 = 0.$$

Here, we have, $V = x^4 - 12x^2 + 12x - 3$; finding the first derivative of V, and suppressing the factor, $+ 4$, we have, $V_1 = x^3 - 6x + 8$; dividing V by V_1, we find, for a remainder, $- 6x^2 + 9x - 8$; suppressing the factor, $+ 3$, and changing the sign of the result, we have, $V_2 = 2x^2 - 3x + 1$; multiplying V_1 by 4, and dividing by V_2, we have, for a remainder, $- 17x + 9$; changing the sign, we have, $V_3 = 17x - 9$; multiplying V_2 by 289, and dividing by V_3, we find, for a remainder, $- 8$; changing the sign, we have, $V_4 = 8$; writing these expressions in a column, and substituting $- \infty$, and then $+ \infty$, for x, we have the results indicated below.

	$x = - \infty$.	$x = + \infty$.
$V = x^4 - 12x^2 + 12x - 3$,	$+$	$+$
$V_1 = x^3 - 6x + 3$,	$-$	$+$
$V_2 = 2x^2 - 3x + 1$,	$+$	$+$
$V_3 = 17x - 9$,	$-$	$+$
$V_4 = 8$,	$+$	$+$

In the first case, there are 4 variations, and in the second, there are no variations; hence, all four of the roots are real.

Sturm's Theorem also enables us to determine the places of the real roots. If we substitute for x, in the above expressions, any two numbers whatever, the number of variations corresponding to the less, diminished by that corresponding to the greater, will give the number of real roots between the two numbers. Let us begin by making x equal to 0, 1, 2, &c., until we get as many permanences, as when $x = \infty$; then make x equal to $- 1$, $- 2$, &c., until we get as many variations, as when $x = -\infty$. Taking the same example as before, we have the following results:

$x =$	$-4,$	$-3,$	$-2,$	$-1,$	$0,$	$1,$	$2,$	$3.$
V,	+	−	−	−	−	−	−	+
V_1,	−	−	+	+	+	−	−	+
V_2,	+	+	+	+	+	0	+	+
V_3,	−	−	−	−	−	+	+	+
V_4,	+	+	+	+	+	+	+	+

Hence, we conclude that two of the roots of the given equation lie between 0 and + 1; one between 2 and 3; and one between − 3 and − 4. Here, we have found the values of the roots, to within less than 1.

In the preceding case, we have seen that + 3 gives the same number of permanences as $+\infty$; hence, no real roots lie between + 3 and $+\infty$; we have also seen, that − 4 and $-\infty$, give the same number of variations; hence, no real root lies between them. The values, − 4 and + 3, are called the **limits** of the roots of the given equation; the former being the *inferior*, and the latter, the *superior* limit.

If we consider the positive roots alone, 0 and 3 are the limits; if we consider the negative roots alone, − 4 and − 3 are the limits. In the same manner, the limits of the positive and negative roots of any equation may be found. It is often useful to determine these limits, especially when seeking the entire roots of an equation, by the process of article 206.

EXAMPLES.

1. Find the number, the places, and the limits of the real roots of the equation, $x^3 + x^2 + x - 100 = 0$.

OPERATION.

$$
\begin{aligned}
V &= x^3 + x^2 + x - 100 \\
V_1 &= 3x^2 + 2x + 1 \\
V_2 &= -4x + 901 \\
V_3 &= -2442627
\end{aligned}
$$

$x =$	$-\infty$,	$+\infty$,	0,	1,	2,	3,	4,	5.
V,	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$+$
V_1,	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
V_2,	$+$	$-$	$+$	$+$	$+$	$+$	$+$	$+$
V_3,	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Hence, there is one real root lying between 4 and 5, which are the limits of the root.

2. Find the number, places, and limits of the real roots of the equation,

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

Ans. There are 4 real roots; one between 0 and 1, one between 2 and 3, one between 5 and 6, and one between -1 and 0. The limits are -1 and 6.

3. Find the number, places, and limits of the real roots of the equation, $x^3 - 23x - 24 = 0$.

Ans. 3 real roots; one between 5 and 6, one between -1 and -2, and one between -4 and -5. The limits are, -5 and $+6$.

4. Find the number, places, and limits of the real roots of the equation, $x^3 + \frac{3}{2}x^2 - 2x - 5 = 0$.

Ans. There is 1 real root, and it lies between the limits 1 and 2.

Each variation is lost when x passes from the preceding value to the root value of V; hence, if the greater number substituted for x is a root of the equation, it is to be counted amongst the roots sought.

CHAPTER XIII.

APPENDIX.

Object proposed.

212. It is proposed to demonstrate, in the following articles, several useful principles, which on account of their difficulty, were omitted from the body of the work. The subjects embraced, are the principles used in factoring, the binomial formula for any exponent, and the summation of series.

Principles used in Factoring.

213. First Principle. *The difference of the like powers of any two quantities, is divisible by the difference of the quantities.*

To demonstrate this principle, let a and b denote any two quantities, and m any positive whole number; then will $a^m - b^m$ denote the difference between the like powers of any two quantities, and $a - b$ the difference between the quantities; if we commence the division by the rule, we shall have the following

OPERATION.

$$\begin{array}{l|l} a^m - b^m & \underline{a - b} \\ \underline{a^m - a^{m-1}b} & a^{m-1} \\ \qquad a^{m-1}b - b^m; & \end{array}$$

the remainder may be factored, and placed under the form,

$$b(a^{m-1} - b^{m-1});$$

hence, we have,

$$\frac{a^m - b^m}{a - b} = a^{m-1} + b\frac{a^{m-1} - b^{m-1}}{a - b} \quad . \quad . \quad (1).$$

The second member of equation (1) will be entire, and consequently, the first member will be entire, when $\frac{a^{m-1} - b^{m-1}}{a - b}$ is entire: that is, *if the difference of the* $(m-1)^{th}$ *powers of two quantities is divisible by the difference of the quantities, then will the difference of the* m^{th} *powers of the two quantities also be divisible by the difference of the quantities.*

But, we know that the difference of the second powers is divisible by the difference of the quantities; hence, from the principle above demonstrated, the difference of the cubes is also divisible by the difference of the quantities; it having been proved that the difference of the cubes is divisible, it follows, from the principle demonstrated, that the difference of the fourth powers is also divisible by the difference of the quantities; the difference of the fourth powers being divisible, it follows, as before, that the difference of the fifth powers is divisible; and so on, by successive deduction, it may be shown that the division is possible when m is any positive number whatever; hence, the principle is proved.

We found the first term of the quotient to be, a^{m-1}; and if we perform a second partial division, we shall get, for the second term of the quotient, $a^{m-2}b$, with a second remainder, $b^2(a^{m-2} - b^{m-2})$; dividing again, we shall find for the the third term of the quotient $a^{m-3}b^2$; and so on. Writing out the quotient, we have,

$$\frac{a^m - b^m}{a - b} = a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \ldots + ab^{m-2} + b^{m-1}. \quad (2).$$

The coefficient of each term of the quotient is equal to 1, and the exponents follow the law explained in deducing the binomial formula (Art. 100).

SECOND PRINCIPLE. *The difference of like even powers of any two quantities is divisible by the difference of the squares of the quantities.*

For, if we replace a by c^2 and b by d^2, equation (2) becomes,

$$\frac{c^{2m} - d^{2m}}{c^2 - d^2} = c^{2m-2} + c^{2m-4}d^2 + \ldots + c^2d^{2m-4} + d^{2m-2} \,. \quad (3).$$

Whatever may be the value of m, $2m$ is an even number, and the second member is entire; hence, the principle is proved.

THIRD PRINCIPLE. *The difference of like even powers of any two quantities, is divisible by the sum of the quantities.*

For, if we multiply both members of (3) by the quantity $(c - d)$, and reduce, we have,

$$\frac{c^{2m} - d^{2m}}{c + d} = (c - d)\,(c^{2m-2} + c^{2m-4}d^2 + \ldots + d^{2m-2}) \,. \quad (4).$$

The second member of (4) is entire; hence, the principle is demonstrated.

Fourth Principle. *The difference of the like powers of two quantities, is divisible by the difference of any other like powers of the two quantities, if the exponent in the first case is divisible by that in the second case.*

If we replace a, in equation (2), by c^n, and b by d^n, n being a whole number, we have,

$$\frac{c^{mn} - d^{mn}}{c^n - d^n} = c^{mn-n} + c^{mn-2n}d^n + \ldots + d^{mn-n} \quad . \quad (5).$$

The second member of (5) is always entire; hence, the principle is demonstrated.

Fifth Principle. *The sum of like odd powers of any two quantities is divisible by the sum of the quantities.*

Let m be any odd number, and let the operation of dividing $a^m + b^m$ by $a + b$ be commenced as shown below:

$$\begin{array}{r|l} a^m + b^m & a + b \\ \cline{2-2} a^m + a^{m-1}b & a^{m-1} \\ \hline -\, a^{m-1}b + b^m & \end{array};$$

factoring the remainder, and writing it over the divisor, as a fraction, we have,

$$\frac{a^m + b^m}{a + b} = a^{m-1} - b\left(\frac{a^{m-1} - b^{m-1}}{a + b}\right) \quad . \quad . \quad (6).$$

Since m is an odd number, $m-1$ is an even number, and consequently the quantity within the brackets is entire, according to the third principle; hence, the proposition is proved.

The form of the quotient is the same as that of $a^m - b^m$ by $a - b$, except that the terms are alternately plus and minus; that is,

$$\frac{a^m + b^m}{a + b} = a^{m-1} - a^{m-2}b + \ldots - ab^{m-2} + b^{m-1} \quad . \; . \quad (7).$$

Value of $(a^m - b^m) \div (a - b)$, when $a = b$.

214. It has been shown that, when m is a positive whole number, we have,

$$\frac{a^m - b^m}{a - b} = a^{m-1} + a^{m-2}b + \ldots + ab^{m-2} + b^{m-1},$$

in which the quotient has m terms. This is true for all values of a and b; hence, it will be true when $a = b$. In this case, $\frac{a^m - b^m}{a - b}$, reduces to $\frac{0}{0}$ in consequence of the existence of a factor in both numerator and denominator, which becomes 0 under the supposition that $a = b$; denoting what the true value of this fraction becomes, when $a = b$, by the symbol, $\left(\frac{a^m - b^m}{a - b}\right)_{a=b}$, and then making $a = b$ in the second member, we have,

$$\left(\frac{a^m - b^m}{a - b}\right)_{a=b} = ma^{m-1} \quad . \; . \; . \quad (1).$$

It may be shown that equation (1) is true whatever may be the value of m; that is, whether m is positive or negative, entire or fractional.

First. Suppose m to be a positive fraction, equal to $\frac{p}{q}$:

Make $a^{\frac{1}{q}} = x$, and $b^{\frac{1}{q}} = y$; whence, $a^{\frac{p}{q}} = x^p$, $b^{\frac{p}{q}} = y^p$, $a = x^q$, and $b = y^q$; we have

$$\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b} = \frac{x^p - y^p}{x^q - y^q};$$

dividing both terms of the second member, by $x - y$, we have,

$$\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b} = \frac{\left(\frac{x^p - y^p}{x - y}\right)}{\left(\frac{x^q - y^q}{x - y}\right)} \quad . \quad . \quad . \quad (2)$$

If we make $a = b$, we have, $x = y$, in which case, the numerator of the second member of (2) becomes, px^{p-1}; and the denominator becomes qx^{q-1}, as shown above; hence,

$$\left(\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b}\right)_{a=b} = \frac{px^{p-1}}{qx^{q-1}} = \frac{p}{q}x^{p-q};$$

or, substituting for x, its value, $a^{\frac{1}{q}}$, and reducing,

$$\left(\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{a - b}\right)_{a=b} = \frac{p}{q}a^{\frac{p}{q}-1}; \quad . \quad . \quad . \quad (3)$$

hence, *equation* (1) *is true for all positive values of the exponent.*

Secondly. Suppose m to be negative, either entire or fractional; that is, let $m = -n$. We shall have,

$$\frac{a^{-n} - b^{-n}}{a - b} = -a^{-n}b^{-n} \times \frac{a^n - b^n}{a - b}, \quad . \quad . \quad (4)$$

because, $a^{-n} - b^{-n} = -a^{-n}b^{-n} \times (a^n - b^n)$.

If $a = b$, the first factor of the second member of (4) reduces to $-a^{-2n}$; and the second factor reduces to $+na^{n-1}$, by the preceding principle; hence,

$$\left(\frac{a^{-n} - b^{-n}}{a - b}\right)_{a=b} = -a^{-2n} \times na^{n-1} = -na^{-n-1}; \quad (5)$$

hence, *equation* (1) *is true when the exponent is negative, either entire or fractional.* It is, therefore, general; that is, the equation,

$$\left(\frac{a^m - b^m}{a - b}\right)_{a=b} = ma^{m-1},$$

is true for all values of m, positive or negative, entire or fractional.

General Demonstration of the Binomial Theorem.

215. The object of the binomial theorem is to show how to develop any power of a binomial into a series.

Let us assume the development,

$$(1 + z)^m = P + Qz + Rz^2 + Sz^3 + \&c. \quad . \quad . \quad (1).$$

It is required to find such values for P, Q, R, &c., as will make equation (1) true for all values of m and z, that is, such values as will make (1) an identical equation, (Art. 180.)

Since equation (1) is to be true for all values of z, it must be so when $z = 0$; making $z = 0$, in (1), we have,

$$1 = P, \quad \text{or} \quad P = 1;$$

substituting this value of P in (1), it becomes,

$$(1 + z)^m = 1 + Qz + Rz^2 + Sz^3 + \&c. \quad . \quad . \quad (2).$$

Since (2) is to be true for all values of z, we may replace z by y, giving,

$$(1 + y)^m = 1 + Qy + Ry^2 + Sy^3 + \&c.; \quad . \quad . \quad (3)$$

subtracting (3) from (2), member from member, we have,

$$(1+z)^m - (1+y)^m = Q(z-y) + R(z^2-y^2) + \&c.; \quad (4)$$

dividing the first member of (4), by $(1 + z) - (1 + y)$, and the second by its equal, $z - y$, we have,

$$\frac{(1+z)^m - (1+y)^m}{(1+z) - (1+y)} = Q + R\frac{z^2-y^2}{z-y} + S\frac{z^3-y^3}{z-y} + \&c. \quad (5).$$

If, in (5), we make $1 + z = 1 + y$, whence $z = y$, in accordance with the principle demonstrated in the preceding article, we shall have,

$$\left(\frac{(1+z)^m - (1+y)^m}{(1+z) - (1+y)}\right)_{1+z=1+y} = m(1+z)^{m-1},$$

$$\left(\frac{z^2-y^2}{z-y}\right)_{z=y} = 2z, \quad \left(\frac{z^3-y^3}{z-y}\right)_{z=y} = 3z^2, \ \&c.;$$

hence, equation (5), under this hypothesis, becomes,

$$m(1 + z)^{m-1} = Q + 2Rz + 3Sz^2 + \&c.; \quad . \quad . \quad (6)$$

multiplying both members of (6), by $1 + z$, it becomes,

$$m(1+z)^m = \begin{array}{r|r|l} Q + 2R & z + 3S & z^2 + \text{\&c.}; \quad . \quad (7) \\ + Q & + 2R & \end{array}$$

multiplying both members of (2), by m, it becomes,

$$m(1+z)^m = m + mQz + mRz^2 + mSz^3 + \text{\&c.} \quad . \quad (8).$$

The first members of (7) and (8) are equal; consequently their second members are equal; hence,

$$m + mQz + mRz^2 + \text{\&c.} = \begin{array}{r|r|l} Q + 2R & z + 3S & z^2 + \text{\&c.} \\ + Q & + 2R & \qquad (9). \end{array}$$

Equation (9) is an identical equation, that is, it is true for all values of m and z; hence, from the principle of indeterminate coefficients (Art. 181), the coefficients of the like powers of z, in the two members, are equal; equating these coefficients, we have,

$$\begin{array}{lll} m = Q, & \therefore & Q = m, \\ mQ = 2R + Q, & \therefore & R = \dfrac{m(m-1)}{1 \, . \, 2}, \\ mR = 3S + 2R, & \therefore & S = \dfrac{m(m-1)(m-2)}{1 \, . \, 2 \, . \, 3}. \end{array}$$

The law of coefficients is apparent.

Substituting these values in (2), we have,

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1 \, . \, 2} z^2 + \frac{m(m-1)(m-2)}{1 \, . \, 2 \, . \, 3} z^3 + \text{\&c.} \quad (10).$$

If we replace z by $\frac{b}{a}$, in (10), and then multiply both members by a^m, and reduce, we have,

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1\,.\,2}a^{m-2}b^2$$
$$+ \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^{m-3}b^3 + \&\text{c.}, \quad (11)$$

which is the binomial formula; it is true for any value of m, either positive or negative, entire or fractional.

By comparing equation (11) with that of (Art. 100), we see that they are of the same form; hence, the formula deduced in Art. 100 is general.

EXAMPLES.

Develop the following expressions:

1. $\sqrt{a-x}$; or, $(a-x)^{\frac{1}{2}}$.

Ans. $\sqrt{a}\left(1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{3x^3}{48a^3} - \frac{15x^4}{384a^4} - \&\text{c.}\right)$.

2. $(a^2 + x^2)^{\frac{3}{2}}$.

Ans. $a^3 + \frac{3ax^2}{2} + \frac{3x^4}{8a} - \frac{3x^6}{48a^3} + \frac{9x^8}{384a^5} - \&\text{c.}$

3. $\frac{a^3}{(a-x)^3}$; or, $a^3 \times (a-x)^{-3}$.

Ans. $1 + \frac{3x}{a} + \frac{6x^2}{a^2} + \frac{10x^3}{a^3} + \&\text{c.}$

4. $\frac{1}{\sqrt{1+x}}$; or, $(1+x)^{-\frac{1}{2}}$.

Ans. $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \&\text{c.}$

5. $(3-7x)^{-\frac{1}{3}}$.

Ans. $\frac{1}{\sqrt[3]{3}}\left(1 + \frac{7x}{9} + \frac{2}{9}\left(\frac{7}{3}\right)^2x^2 + \frac{14}{81}\left(\frac{7}{3}\right)^3x^3 + \&\text{c.}\right)$

Summation of Series.

216. The **summation** of a series is the operation of finding an expression for the sum of any number of terms of the series. The method of greatest general utility, is that by **differences.** Let a, b, c, d, &c., represent the successive terms of any series. If we subtract each term of this series from the following one, we shall thus form a new series, which is called the **first order of differences.** If, in like manner, we subtract each term of this series from the following one, we shall form a new series, called the **second order of differences**; and so on, as exhibited below:

$$\begin{array}{llllll} a, & b, & c, & d, & e, & \text{\&c.} \\ & b-a, & c-b, & d-c, & e-d, & \textit{1st order of diff.} \\ & c-2b+a, & d-2c+b, & e-2d+c, & & \textit{2d order of diff.} \\ & & \text{\&c.,} & \text{\&c.} & & \end{array}$$

If we designate the first terms of the first, second, &c., orders of differences by d_1, d_2, &c., we shall have,

$$\begin{array}{lll} d_1 = b - a & \therefore & b = a + d_1 \\ d_2 = c - 2b + a & \therefore & c = a + 2d_1 + d_2 \\ d_3 = d - 3c + 3b - a & \therefore & d = a + 3d_1 + 3d_2 + d_3 \\ \qquad \text{\&c.,} & & \qquad \text{\&c.} \end{array}$$

If we designate that term of the given series which has n terms before it, by T, we shall find, by continuing the operation,

$$T = a + nd_1 + \frac{n(n-1)}{1.2}d_2 + \frac{n(n-1)(n-2)}{1.2.3}d_3 + \text{\&c.} \quad (1).$$

This formula enables us to find the $(n+1)^{th}$ term, when we know the first term of the series, and a sufficient number of the first terms of the successive orders of differences.

Let us take the series,

$$0, \ a, \ a+b, \ a+b+c, \ a+b+c+d, \ \&c., \ . \ . \quad (2).$$

The first order of differences of series (2) is the series,

$$a, \ b, \ c, \ d, \ \&c., \ . \ . \ . \ . \ . \quad (3).$$

Now, it is clear that the sum of n terms of series (3) is equal to the $(n+1)^{th}$ term of series (2); but the first term of the first order of differences is a, the first term of the second order of differences is d_1, and so on. If, therefore, we denote the sum of n terms of series (3), which is the same thing as the $(n+1)^{th}$ term of (2), by S, we shall have its value from (1), by making $a = 0$, $d_1 = a$, $d_2 = d_1$, $d_3 = d_2$, &c.; hence,

$$S = na + \frac{n(n-1)}{1 \, . \, 2} d_1 + \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3} d_2$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1 \, . \, 2 \, . \, 3 \, . \, 4} d_3 + \&c. \quad (4).$$

If all the terms of any order of differences become equal, each term of every succeeding order of differences becomes 0, and the formula gives the exact sum of the terms. If the terms of no order of differences become equal, the formula only gives approximate results, which will be nearer the true sum, the greater the number of terms employed.

EXAMPLES.

1. Find the sum of n terms of the series,

$$1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2,\ \&c.$$

OPERATION.

Series,	1,	4,	9,	16,	25,	36,	&c.
1st order of differences,	3,	5,	7,	9,	11,	&c.	
2d order of differences,	2,	2,	2,	2,	&c.		
3d order of differences,	0,	0,	0,	&c.			

Here, $a = 1,\ d_1 = 3,\ d_2 = 2,\ d_3 = 0,\ d_4 = 0,$ &c.;

hence, by substitution in (4),

$$S = n + \frac{n(n-1)}{1\,.\,2} \times 3 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} \times 2.$$

2. Find the sum of 8 terms of the series,

$$1,\ 1 + 2,\ 1 + 2 + 3,\ 1 + 2 + 3 + 4,\ \&c.$$

The first order of differences is,

2, 3, 4, &c.,

the second order of differences is,

1, 1, 1, &c.,

and succeeding orders are made up of zeros.

Here, $n = 8,\ a = 1,\ d_1 = 2,\ d_2 = 1,\ d_3 = 0,$ &c.; hence,

$$S = 8 + \frac{8 \times 7}{1\,.\,2} \times 2 + \frac{8 \times 7 \times 6}{1\,.\,2\,.\,3} \times 1 = 8 + 56 + 56 = 120.$$

3. Find the sum of 10 terms of the series,

$1^3,\ 2^3,\ 3^3,\ 4^3,\ 5^3,$ &c. *Ans.* $S = 3025$

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For use in district schools, and for supplying a text-book in advanced work for classes having finished the course as given in the ordinary Practical Arithmetics, the National Arithmetic has been divided and bound separately, as follows: —

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This work, as its title indicates, is elementary in its character and suitable for use (1) in such public schools as give instruction in the Elements of Algebra: (2) in institutions of learning whose courses of study do not include Higher Algebra; (3) in schools whose object is to prepare students for entrance into our colleges and universities. This book will also meet the wants of students of Physics who require some knowledge of

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